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DISCOVERY TEACHING AND PROBLEM SOLVING IN SENIOR HIGH SCHOOL MATHEMATICS

by

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ABSTRACT

The purpose of the study was to develop treatments which emphasize certain aspects of discovery teaching and to test these treatments in their ability to improve problem solving while maintaining achievement in mathematics teaching. Thirty six Mathematics 10 students were divided into three treatment groups of twelve students each. All three treatments, taught by the researcher, lasted eleven days, eighty minutes per day, with two days of testing for problem solving and achievement.

The three aspects of discovery teaching examined were: 1) Student interaction, with students working independently as an expository aspect and students working in groups as a discovery aspect, 2) Teacher guidance, with the teacher controlling the classroom learning completely as an expository aspect and the teacher only introducing problems and consolidating the learning as a discovery aspect, and 3) Integration, with practice exercises like those studied as an expository aspect and problems involving new hypotheses as a discovery aspect. Three treatments were developed: 1) Expository (E), containing all expository aspects, 2) Discovery I (D1), containing discovery aspects in student interaction and teacher guidance and an expository aspect in integration, and 3) Discovery II (D2), containing all discovery aspects. The Student Inventory of Teacher Behavior, involving a five-point preference scale on thirty items testing teacher omniscience, introduction of generalization, control of pupil interaction, methods of answering questions, use of student responses, and method of eliminating false concepts, was given to the students following the treatments. T-test results indicate a significant difference at .05 level between treatments.

To test problem solving, the researcher developed the Problem Solving in Rational Expressions Test, consisting of eight items defined as problems and involving rational expressions in the solutions. Four scoring schemes were developed and applied, with separate statistical analyses reported: 1) Correct Answer, awarding points only to the correct answer, or product, 2) Quality of Answers, awarding full points only to the correct answer and awarding partial points to partial results, 3) Polya's Four Phases of Problem Solving, awarding points at all stages of problem solving, understanding the problem, design, procedure, and solution, and 4) Quality of Response Approaches, awarding points for process of problem solving. There was no significant difference using ANOVA at .05 level between Groups E, D1, and D2 for scoring schemes 1, 2, and 3, but there were significant differences at .05 level using a t-test between E and D2 for scoring scheme 3, and between the groups using ANOVA for scoring scheme 4 in favor of D2.

To test achievement, the researcher developed the Achievement in Rational Expressions Test, containing forty items similar to the practice exercises of E and D1, covering the topics of the unit. ANOVA results indicate no significant difference at .05 level between groups.

The researcher concludes that problem solving processes can be improved using the discovery method involving problems in the integration phase, while maintaining ability to find the product in problem solving and achievement.

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Chapter I

INTRODUCTION TO THE PROBLEM

Background and Importance of the Problem

Many authors have addressed themselves to the need for problem solving in the curriculum. Futurists agree that the coming curriculum needs to be based more upon process, in particular problem solving, discovery, experimentation, and evaluation, and less upon content. (Baughman 1971) "The future will bring new and more complex problems. Their solution will depend on whether society recognizes the need for training people in problem solving efforts." (Weir 1974, p. 16) Crutchfield expands this idea by the contention that we cannot know the specific facts or even skills to teach in light of the knowledge explosion and rapidly changing state of the world. (Crutchfield 1965) Shapiro (1974) adds:

> Creative problem solving--the processing and manipulation of information for affective and cognifive goals--is believed by many educators to be the most significant skill needed by children to cope with a future society. (p. 42)

Bruner (1966) looks for an approach to learning which allows the child to learn the material presented in such a way that he can use the information in problem solving. "We teach a subject, not to produce little living libraries from that subject, but rather to get a student to think for himself, to take part in the process of knowledge-getting." (Bruner 1966, p. 72)

Other authors have addresses themselves more specifically to the need of problem solving in the mathematics curriculum. The National

Council of Supervisors of Mathematics (1977) takes the position that problem solving is the principal reason for studying mathematics. In addition, the United States Conference Board of the Mathematical Sciences states at the International Congress of Mathematicians:

> We regard problem solving as the basic mathematical activity. Other mathematical activities such as generalization, abstraction, theory building, and concept formation are based on problem solving. (Rosenbloom 1966, p. 130)

Decision-making in our society involves the ability to think rationally and objectively. Historically, this has been the objective of mathematics teaching, which provides experiences in gathering and analyzing information and making deductions from that information. (Cooney 1975) Troutman and Lichtenburg (1974) see mathematics problem solving as critical to real life situations. After leaving the classroom, students encounter situations that can be interpreted from the structure of a mathematical model. Polya (1965) also feels that one of the principal aims of the high school mathematics curriculum is to develop the students' ability to solve problems. Teaching problem solving in the mathematics class benefits all students, whatever their future occupation may be. There is the opportunity to develop certain concepts and habits of mind, which are important ingredients of the general culture. "Problem solving qualifies as the ultimate justification for teaching mathematics ... Ultimately, a student learns mathematics in order to solve a great variety of problems." (Travers et al 1977, p. 121)

The discovery method of teaching and learning is also cited in the literature as an important part of the curriculum. According to Dewey (1933):

> for the teacher or book to cram pupils with facts which, with little more trouble, they could discover by direct inquiry, is to violate their intellectual integrity and to cultivate mental servility. (p. 257)

Young (1971) believes that the effort of discovery is important, successful or not. Samples (1968) agrees that only through the process of discovery will a student learn to make discoveries. Children should be provided with problems that give them opportunities to make discoveries that are within their reach. (Adler 1971) According to Bruner (1971):

> With the active attitude that an emphasis on discovery can stimulate, with greater emphasis on intuition in our students, and with a courteous and ingenious effort to translate organizing ideas into the available thought form of our students, we are in a position to construct curricula that have continuity and depth and that carry their own reward in giving a sense of increasing mastery over powerful ideas and concepts that are worth knowing... (p. 177)

Taba (1965) calls learning by discovery the "chief mode for intellectual productivity and autonomy." The student, through discovery, has increased ability to learn in unknown areas, to gather data, and to abstract from new ideas and concepts.

"The role of any item of content or of procedure depends upon two things, its potential value with reference to the goals of mathematics instruction, and the effectiveness with which the item is incorporated into the classroom teaching process." (Jones 1966, p. 106) Since problem solving in the mathematics curriculum and in the curriculum in general seems to be such an important goal, an effective method of teaching problem solving must be found. A single unit on problem solving does not seem to be the answer. Problem solving needs to be used continually in all units of mathematics. One way to accomplish

this goal is to use a discovery approach to teaching mathematics. As shown, discovery teaching is also considered an important part of the curriculum. "The use of discovery in teaching to build the students' confidence in their own ability to think and solve problems is certainly a highly desirable goal." (Morrisett 1966, p. 178) S. Williams (1975) sees problem solving skills and creative discoveries both encouraged in the classroom. When children learn by discovery, they are able to generalize their skills to solve problems that exist outside the classroom. (Kagan 1966) Bruner feels that discovery in learning is a necessary condition for learning the variety of techniques of problem solving. "Practice in discovering for oneself teaches one to acquire information in a way that makes that information more readily viable in problem solving." (Bruner 1968, p. 164)

Law feels that teachers place too much emphasis on skills, or practicing a routine, and that once a technique has been demonstrated and practiced, the student is no longer in a problem solving situation. He fears that repetitive exercises will result in a reduction of the pupil's ability to cope in a new situation. He looks for the increase in discovery methods to encourage a problem solving approach. (Law 1972) To sum up the importance of problem solving through discovery, Polya (1957) has to say:

> A great discovery solves a great problem, but there is a grain of discovery in the solution of any problem. Your problem may be modest; but if it challenges your curiosity and brings into play your inventive faculties, and if you solve it by your own means, you may experience the tension and enjoy the triumph of discovery. Such experiences at a susceptible age may create a taste for mental work and leave their imprint on mind and character for a lifetime. (p. v)

The Statement of the Problem

The problem is to develop treatments which emphasize certain aspects of discovery teaching and to test these treatments in their ability to improve problem solving while maintaining achievement.

Research Questions

1. Can expository and discovery treatments be developed on the same topic unit such that students will be able to distinguish between them?
2. Are there differences in achievement between students taught by a treatment involving no aspects of discovery teaching (expository) and treatments involving aspects of discovery teaching?
3. Are there differences in problem solving ability between students taught by an expository treatment and discovery treatments?

Definitions

1. Discovery teaching is a teaching strategy which involves: 1) an absence of teacher guidance except to present and consolidate a problem, 2) students working together in pairs, small groups, or as a class to explore, hypothesize, and evaluate a problem, and 3) integrating the processes of solving problems.
2. Expository teaching is a teaching strategy which involves: 1) total teacher guidance, 2) students working individually,

and 3) practicing previously learned algorithms.

3. Problems are situations or questions which contain an obstacle or challenge that cannot be overcome by any automatic or previously learned algorithm.
4. Problem solving is the process of accepting the challenge of a problem and striving to resolve it.
5. Product of problem solving is the answer to the problem.
6. Process of problem solving is the approach used to solve the problem.

Chapter II

REVIEW OF THE LITERATURE

Theories of Teaching Problem Solving and Related Research

Different theories of teaching and improving problem solving ability have been offered. Kinsella (1970) gives the following suggestions for improving problem solving: 1) Before presenting problems, provide for recall and organization of concepts, generalizations, theorems, and methods relevant to the problems to be presented, 2) Study the relations among the elements of problems and find the connections between what is given and what is found, and 3) Praise different ways of solving problems, encourage easier and more direct ways of solving problems, have the students try one way to solve different kinds of problems, and use open-ended problems. Encourage students to be alert to neat solutions and alternative solutions. Point out that some solutions are more aesthetic than others. Although these suggentions are not contradictory, they contain different viewpoints.

The first of Kinsella's theories deals with a hierarchy of knowledge, as Gagne's emphasis on sequences and organization of learning. For Gagne, the act of problem solving is the highest level of mathematical endeavor, and comes after the content learning of facts, concepts, and principles. (Shulman 1973) "To be an effective problem solver, the individual must somehow have acquired masses of structurally organized knowledge. Such knowledge is made up of content principles, not heuristic ones." (Gagne 1965, p. 170) Staats (1966) agrees that

in order to produce orginal problem solvers, the best experience is a good education in the field containing the problems. Fine (1966) adds that the time spent in teaching problem solving skills would be better used in the systematic study of some important discipline. "Although the development of problem solving ability as an end in itself is a legitimate objective of education, it is less central an objective than that of learning the subject matter." (Ausubel 1973, p. 151)

The application of instructional planning techniques (as identification and prescription of specific, behaviorally stated objectives, the diagnosis of characteristics and previous learning of students, the utilization of carefully programmed and sequential instructional activities, and frequent evaluation) to creative problem solving courses can result in productive benefits to the students. (Treffinger 1975) Brownell advocates problem solving based on understanding and gives preference to the form of instruction which enables the learner to best organize previous learning. (Brownell 1942)

Maier (1930) studied the role of direction on reasoning and problem solving. Providing additional information resulted in improving the problem solving process. (Wittrock 1966) Scandura (1966) found that prerequisite practice, with feedback, facilitated problem solving. Exposure to information about the problems, together with related practice, reliably improved problem solving performance.

This method of improving problem solving emphasizes teaching accumulated facts, concepts, and principles as needed prior to problem solving.

The second method of improving problem solving as suggested by Kinsella is an analysis of the act of problem solving, making students

aware of the processes involved in problem solving. Dewey, Polya, and other advocates of step-by step heuristics fall into this category of teaching problem solving. Students are given a list of questions to ask about the problem or a specific method of attacking the problem.

Post (1967) gave six weeks of practice on a list of processes from the literature on problem solving to ten Grade VII mathematics classes. He found no significant difference between the experimental and control groups. Brian (1967) also found no improvement in ability to construct mathematical models, to conjecture, to use axioms, theorems and algorithms, but some improvement in ability to settle conjectures, among college students trained in problem solving using a flow chart taken from Polya's heuristics. Similar results from Lerch (1966) using Grade V students, age ten to twelve, in a structured equation approach for the experimental group of 28 students and a traditional approach of having students ask questions about the probelm in the control group. Neither group grew more in complete problem solving ability, but the experimental group grew more in determining the procedure to use in solving problems.

Wilson (1967) used eighty Grade IV students in three groups to test for choice of correct operation in problem solving. The three groups differed in kind of structure: 1) Action-sequence, 2) Wanted-given, and 3) Control with practice only. After nine weeks of instruction, three hours per week, the wanted-given group performed best on correct operation choice on one-step verbal problems.

Smith (1973) found that subjects who received general heuristic instruction did not solve more transfer problems and did not solve them faster than did the subjects who received task-specific heuristic

instruction. Task-specific heuristic instruction is more effective that general heuristic instruction in improving problem solving performance on some learning tasks.

Yates (1976) trained students in a method adapted from Polya's heuristics. Unlike Brian, who had no significant results, Yates found that the training improved significantly the problem solving of persons not noted for creativity. The abilities to deduce, criticize, and organize were improved, while abilities to imagine and invent were not improved.

Crutchfield (1965) also had positive results in training for problem solving. Upper elementary students were given lessons individually, each lesson a complete problem solving episode containing all the principal steps and processes in creative problem solving. The experimental group showed a marked superiority in creative problem solving over the control group, but not when the test problems were of a purely logical nature.

Covington (1968) used a general problem solving program of sixteen lessons for Grades V and VI, each lesson a mysterious occurence or unexplainable happening which gave training in creative thinking and problem solving. The trained group involved 267 students, and 214 were in the control group. The performance of instructed children was significantly better than the control group on tests of problem solving and creative thinking. A retention test five months later also resulted in the trained group scoring significantly better than the control group in problem solving, but not creativity.

Keil (1964) used a method of improving problem solving which seems different from the ones Kinsella recommends. The Grade VI students who

wrote and solved problems of their own were superior in arithmetic problem solving ability to the Grade VI students who solved textbook problems. Two hundred children were used in the experiment. The treatment lasted sixteen weeks, one day per week, while the rest of the time students used the textbook.

Treffinger (1971) also seemed to use a different method from Kinsella's three. He used 739 pupils in thirty six Grade V classes in three treatment groups: 1) Discussion treatments in which the teacher interacted with the pupils, discussing materials, and attempting to teach creative thinking and problem solving, 2) Non-discussion treatments in which the teachers just distributed materials and supervised, and 3) Control group with normal classroom activities. He concluded that creative thinking and problem solving abilities of Grade V pupils can be influenced by instructional efforts.

Kinsella's third method of improving problem solving ability suggests discovery teaching.

Theories of Discovery Teaching

The meaning of discovery teaching varies. Morrisett (1966), Weimer (1975), Davis (1973), and Glaser (1966) believe that it changes from discipline to discipline. Strike (1975) adds that the meaning also differs with the level of difficulty within a discipline. There are two kinds of skills involved in the act of discovery: 1) heuristic skills involving the formulation of hypotheses and 2) epistemic skills involving the verification of hypotheses. The different heuristic and epistemic rules are different between disciplines and levels of difficulty within a discipline. When considering the variables involved in

the concept of discovery, the verification skills are central. (Strike 1975)

Weimer (1975) also believes there is no one common use of the term discovery. His definition of discovery is a teaching strategy in which the object of the lesson is not explicitly communicated to the learner. He has a taxonomy of discovery teaching: 1) Inductive, guided or unguided, 2) Semi-inductive, 3) Deductive, guided or unguided, and 4) Unguided or pure discovery, where only the problem is given.

Gagne's position on discovery is an intervening process in which the student is not told an answer and finds it for himself. (Shulman and Keislar 1966)

Higgins' (1971) characteristics of discovery teaching are: 1) approaching the content through problems, 2) reflecting the problem solving techniques in the logical construction of instructional prodecures, 3) demanding flexibility for uncertainty and alternate approaches, and 4) maximizing student action and participation in the teaching-learning process.

The common feature of theories of discovery is the change in roles of teacher and students. The students actively participate to discover the object of the lesson, while the teacher presents the problem to the students, but does not explicitly communicate the object of the lesson.

Research on Discovery Teaching

Breaux (1975) found that in inductive reasoning, the discovery method had the highest mean error on a transfer task than either deductive reasoning from discovery or presentation or inductive by presentation. In a guided-discovery method versus a conventional

method in Grade IX General Mathematics involving 290 students in twelve classes, the conventional group performed significantly better on one achievement test (GMAT) but not on the other (STEP). There was no significant difference in change of attitude. (Howitz 1965) In attempting to determine the relationship of the discovery method in mathematics to creative thinking, Studer (1971) studied Grade IV and VI students in expository and discovery treatments. Using the Torrence Tests of Creative Thinking, which measure fluency, flexibility, originality, and elaboration, the Grade VI expository and Grade IV discovery were more creative.

Roughead and Scandura (1968) were concerned with two questions: 1) Can "what is learned" in mathematical discovery be identified and, if so, can it be taught by exposition with equivalent results? and 2) How does "what is learned" depend on prior learning and on the nature of the discovery treatment itself? The discovery subjects had to learn to derive solutions whereas solution-given subjects did not. The answer to question (1) is that sometimes "what is learned" during guided discovery can sometimes be identified and taught by exposition. However, the discovery subjects may have attained a higher order ability to derive rules for discovery. In question (2), the presentation order is critical when the hints provided during discovery are specific to formulae rather than a general strategy, and the presentation is not critical when the subjects learn derivation rules.

Gagne and Brown (1961) also were looking for "what is learned," but were unable to find the differences between the superior discovery groups and the rule-given groups. High school students were placed in three experimental groups: Rule and Example, Discovery, and

Guided Discovery. In transfer to new rules, the guided discovery group had the best performance, then the discovery group, with rule and example the worst. The results support a guided discovery procedure for the teaching of the derivation of new but related rules.

Another study on guided discovery with results in favor of guided discovery involved 135 Grade IX boys studying the micrometer caliper. The three groups were traditional, discovery, and control. On initial learning and retention at one week, there was no significant difference between traditional and discovery. However, there were significant differences in favor of discovery in retention at six weeks, transfer at one week, and transfer at six weeks. (Ray 1961)

In a research study with Grade X General Mathematics students in three groups, traditional, discovery, and transfer to problems of the real world, Price (1967) found that the discovery group showed a greater increase in mathematical reasoning than the control group, and the transfer group showed significant increase in critical thinking ability.

Jordy (1976) had mixed results with two programs, Unified Mathematics II for Grade VIII and Unified Mathematics III for Grade IX. The experimental groups had ten discovery lessons, and the control groups used the textbook. In Grade VIII, the experimental group was slightly lower in achievement and had a significantly positive attitude. The groups were equal on creativity. The experimental group in Grade IX had 75% of items correct on the achievement test, while the control group had 50% of items correct on the achievement test. There was no effect on attitudes.

Guthrie (1967), Hermann (1969), and Rizzuto (1970) found fewer

errors on a transfer test for the discovery group than an expository group.

A descriptive study on one mode of discovery teaching, the Mathematizing Mode, was done by Johnston (1968). He gives four stages: 1) Introduction of the Activity, during which the teacher presents the problem situation and the students explore relationships, patterns, and possible solutions, 2) Brainstorming Session, during which the students share all hypotheses generated with the group and the teacher records the hypotheses without attempting to evaluate them, 3) Seminar Type Discussion, during which the teacher introduces the necessary conventions and definitions, the students evaluate the hypotheses through discussion, and the teacher provides questions for practice to reinforce the established hypotheses and to explore for the next activity, and 4) Summary, during which the teacher introduces the precise mathematical description of the ideas already discussed, and the students apply the mathematical principles to problems. Johnston then applied this mode of teaching to specific units and described the results.

Tschofen (1973) did a follow-up descriptive study on the Mathematizing Mode. She also applied the mode of teaching to a specific unit and described the implementation in the classroom. In trying to classify her observations into Johnston's (1968) four stages, she arrived instead at six activities which take place during a mathematizing unit: 1) Introduction, during which the unit is introduced briefly by the teacher, and the students are presented with the problem, 2) Exploration of the Problem during which the students redefine, reorganize, and uncover existing relationships of the problem, 3) Hypothesizing, during which students consider possible solutions or

relationships and present them to the class, without any effort at evaluation, 4) Evaluation, during which students check ideas against mathematical reality, and hypotheses are accepted, rejected, or altered, 5) Summary, during which the teacher and students summarize the concepts and relationships, verbalize the generalizations, and make known the conventions, and 6) Practice, during which students practice techniques and concepts they have learned, either after the summary or during all activities, especially evaluation. All activities are accomplished through a personal-inquiry session or a group discussion.

Sigurdson (1971) lists four phases of the Mathematizing Mode: 1) Exploration, 2) Hypothesizing, 3) Evaluation, and 4) Closure, standardizing, consolidating, and practice. Students work in pairs when not involved in a large group discussion by the entire class. The emphasis is on process objectives as well as content objectives. To this end, all hypotheses are tolerated, correct or incorrect, and all students are actively involved. An open thinking atmosphere is encouraged.

The last three researchers focused on student discovery and evaluation, rather than teacher exposition. The teacher presents the problem and consolidates the learning students have been actively involved in.

Theory Supporting the Use of Discovery to Teach Problem Solving

Bruner (1968) believes that discovery in learning is a necessary condition for learning the variety of techniques of problem solving. "Problem solving situations are usually designed to require discovery on the part of the learner. This is an inevitable part of their makeup;

otherwise they would probably not be called problem solving." (Gagne 1966, p. 147) Troutman and Lichtenberg (1974), in describing the specific abilities related to problem solving, say that the teacher helps students discover properties of mathematical models when they find characteristics of objects or mathematical ideas. Strickland (1968) reports a surprising sophistication and ability to generalize by almost all students, regardless of apparent academic ability, when science teachers have used a discovery approach to encourage critical thinking and problem solving. Gray (1975), also, in describing a program where students use an inquiry method, claims that "this would seem to link aspects of problem solving and creativity with a discovery method." (p. 244)

Even Ausubel, who is generally considered to be an opponent of discovery, "challenges discovery teaching and learning as a unique generator and developer of motivation and problem solving ability." (Ausubel 1973, p. 233) By his use of the word "unique," he implies the relationship between discovery teaching and problem solving ability.

Although the following articles do not directly refer to the connection between discovery and problem solving, the underlying assumption is that through efforts of problem solving one is learning by discovery. According to Craig (1956):

> Many have advocated relatively independent problem solving in the belief that learning situations should be similar to anticipated transfer situations. This point of view rests on the assumption that future discovery of principles will be through independent problem solving, hence it is more like pupil self-discovery than directed discovery... The more direction of this kind available to the learner, the more effective his discovery on new relations. (p. 233)

Anderson (1967) makes a similar point that to develop effective problem solving skills, the student must be trained, have practice, and have an appreciation of the value of being a problem solver. Polya (1957), Brownell (1942), Krulik (1977), Adler (1971), Scandura (1964), and Wittrock (1966) also believe that students learn to solve problems by being actively involved in a problem solving situation.

Bruner (1968) states:

> It is my hunch that it is only through the exercise of problem solving and the effort of discovery that one learns the working heuristics of discovery, and the more one has practice, the more likely is one to generalize what one has learned into a style of problem solving or inquiry that serves for any kind of task one may encounter, or almost any kind of task. (p. 169)

Kagan supports Bruner, saying that when children learn by discovery, they are able to generalize their skills to solve problems that exist outside the classroom.

Research Supporting the Use of Discovery to Teach Problem Solving

In Kersh's (1958) work with high school students, he concluded that the discovery group produced the best performance on the acceptable methods of solving problems approximately one month after learning.

Ninth grade algebra students taught by a discovery method showed significantly greater improvement in problem solving performance than a class taught by a textbook method. (Ashton 1962)

Scandura's (1964) three studies showed improved problem solving ability. Eventhough the treatments were only one day in length, the indirect discovery instruction may have promoted learning how to attack

problems.

Forty five high ability Grade VIII and IX students studying number-sequence concepts by programmed materials in a one-day instruction were the subjects of a study by Meconi (1967). They were divided into three treatment groups: Rule and example, Guided discovery, and Pure discovery. While all three approaches led to learning and there were no significant differences on a problem solving test of immediate transfer, the pure discovery group was the most effective as far a time taken to learn and solve problems.

Wills (1967) determined that students can significantly improve their problem solving ability in addition to content learned by discovery. The subjects were 561 high school students from twenty four Intermediate Algebra classes. Two groups received two weeks of instruction: Covert group was administered instructional materials, and Overt group used instructional materials and received instructions from their teachers. The students in the covert group did as well on the test designed to measure ability on "how to discover" as the group which was taught how to discover.

Worthen (1968) studied Grade V and VI students in the content of integers, the distributive principle, and the exponential notation. The aspect of discovery examined was the sequencing of the controlled variable. Treatment D(discovery) delayed the verbalization of the concept generalization until the end of the instructional sequence, while treatment E(expository) had the verbalization as the initial step in the instruction. Treatment D was significantly better (p .05) on a written heuristic transfer test and (p .025) on an oral heuristic transfer test. Therefore, learning by discovery techniques

significantly increases pupil ability to use discovery problem solving approaches in new situations. However, when the class mean is used as the unit of analysis rather than the pupil as a unit of analysis, the written heuristic transfer test had $F=2.02$ and the oral $F=2.29$. Therefore, there was no significant difference between the treatments.

Sage (1971) found a significant difference in favor of the inquiry method in problem solving performance. Fifty three college students enrolled in three sections of an industrial technical education course received eleven weeks of instruction, one section receiving inquiry training and two sections receiving lectures. Although there was no significant difference on knowledge, a problem solving cognition test, or problem solving time, there was a significant difference on problem solving performance and number of inquiries made in favor of the inquiry trained group.

Williams (1972) also used an inquiry method, in this case with Grade VI students of middle-reading ability, on mathematical verbal problems. Three treatments were used: Inquiry, Taped formal-analysis, and Conventional. The students who received instruction in the inquiry method made significant improvement over the conventional classroom method group in mathematical verbal problem interpretation.

The purpose of Leggette's (1974) study was to determine whether a greater increase in problem solving ability of college freshmen occurs when students are taught to use discovery techniques. The experimental group had three hours per week instruction in the problem solving process as a concept on the use of heuristics in a mathematics class. The problem solving skills of the experimental group increased more than the control group.

Volchansky (1968) worked with fifty six Grade VIII mathematics students, thirty in the control group and twenty six in the discovery group. The students who were taught a mathematical concept through a discovery approach did significantly better at .01 level in answering questions of an analytical nature, a similar process as problem solving.

Ballew (1965) looked at the effectiveness of discovery teaching on critical thinking ability, another similar process to problem solving. He defines critical thinking as a matter of interpreting facts, applying generalizations, and reorganizing errors in logic. He used first year algebra students in two experimental groups and one control group. Both experimental classes improved significantly in critical thinking ability, with no significant difference in achievement between the control class.

Theory and Research Using Groups in Discovery Teaching

Higgins (1971) believes that group processes should be used in discovery teaching. Students working together in a small group to solve a problem may result in much more involvement than struggling to solve the problem individually. Goodlad (1965), Cronbach (1966), Shulman and Keislar (1966), and Williams (1975) also recommend small group work in the study of discovery teaching.

The results of a study by Gagne and Smith (1962) lead them to conclude that verbalization, which indicates group problem solving, helps students think of new responses for their moves in problem solving, and thus facilitates both the discovery of general principles and solving successive problems.

Davidson (1970) studied freshmen college students in Calculus in a guided-discovery approach using small groups. The four-member groups stated and proved theorems of calculus and developed techniques for solving classes of problems. The teacher gave hints, checked solutions, and provided mathematical input in the form of questions for investigation as well as encouragement. The discovery small-group method resulted in slightly better achievement scores than the control group. Additionally, there was a positive or non-negative effect upon each student's interest and estimate of problem solving skill.

The purpose of a study by O'Brien and Shapiro (1977) was to compare discovery learning in groups and individually with reception learning through the use of addition and multiplication grids as vehicles for developing pattern-seeking abilities. The number of patterns found in the grids on the initial achievement, retention, and transfer tests are the dependent variables. The subjects were forty five full-time teachers in an introductory research course required for the master's degree. Three groups of fifteen subjects each were given one of three treatments: Group discovery, Individual discovery, and Reception learning. The group-discovery group performed significantly better on retention and transfer, while the reception group performed significantly better on initial acheivement.

Olah (1976) also studied adults for the effect of group processes on problem solving ability. Two groups were compared in response to three problem solving exercises: 1) Traditional model, which featured a structured educational format, uni-directional cummunication pattern, and instructor dominance over in-class activity, and 2) Innovative model, which featured open discussion, two-way communication, and

participant controlled in-class activities. The participants in the Innovative model responded appropriately to the three problem solving exercises more frequently than the participants in the Traditional model.

Hudgins and Smith (1966) examined the productivity of small problem solving groups of three students in Grades V to VIII. When the most able member of the group is of high status, the number of problems solved cooperatively is no greater than the number of problems solved by the most able member. When the highest ability student is of low status, the group is more productive at solving problems than the individual high ability student.

The evidence from these research studies suggests the most effective discovery teaching methods involve group discoveries to problems.

Theory of Using Practice in Discovery Teaching

According to Kersh (1964), what is learned by the discovery process is what is practiced during the process. Kersh says:

> The learner may acquire more effective ways of problem solving through the discovery process than through another process simply because he has an opportunity to practice different techniques and because his more effective techniques are reinforced. (p. 321)

Chambers (1971) also agrees that the process of discovering principles is probably best learned by practice.

Using discovery methods, then, would be more valuable to students than practicing the concepts or generalizations discovered.

Summary and Implications

The research presented has shown that problem solving can be taught, and one of the methods used is discovery teaching. In discovery teaching, student participation in discovering the object of the lesson without teacher guidance, group work, and the process of making more discoveries are important features of discovery experiences.

Chapter III

CONCEPTUAL FRAMEWORK

Dimensions of Discovery Teaching

The term "discovery" cannot characterize adequately a type of instruction. The term has been associated with any number or combination of approaches. Research in the area of discovery teaching needs better definition. "One obvious prerequisite to further progress in this area is the replacement of such general terms as discovery and exposition with far more precise descriptors." (Shulman 1970, p. 68) Shulman discusses the dimensions along which instructional types can be classified.

The first dimension is degree of guidance. The two extremes are total teacher guidance over the learning situation and no guidance from the teacher.

The second dimension is the sequence of instruction, referring to inductive sequences or deductive sequences. Inductive sequences begin with specific examples or instances followed by a generalization of the rule or principle which can be inferred from the examples. Deductive sequences begin with principles previously known. Additional knowledge is then derived from them.

A third dimension is the contrast between a situation in which a problem is given to students, called problem solving, and a situation in which the learner must formulate a problem, called inquiry.

The fourth dimension under consideration is the didactic-Socratic dimension. Didactic refers to imparting facts rather than inspiring

ideas, with an emphasis on rules, principles, standards of conduct, and authoritative guidelines. Socratic method involves a dialogue between the teacher and student in which the student reaches the desired conclusion through a carefully arranged sequence of questions. The main feature of this dimension is the involvement of the learner.

Four other dimensions along which instruction can be classified are given by Cronbach (1966). The fifth dimension is the character of the guidance in the form of hints given or not given about the solution of the problem.

The sixth dimension is the verbalized or nonverbalized rules. Once a generalization is found, the student can formulate the generalization in words, or he can use it without clearly stating it.

A closely related seventh dimension is the criterion for deciding when a student has reached a solution and can go on to new work. This dimension deals with the consolidation or practice aspect of a learning situation.

The eighth dimension is individual versus group instruction. In group instruction, students help each other face a problem together and throw their partial insights into the discussion.

Three Aspects of Discovery Teaching In Detail

Since a study which considers each of the eight dimensions of discovery teaching would become unwieldy, this study will focus on three aspects of discovery teaching: 1) Student interaction as in dimension eight, 2) Guidance as in dimension one, and 3) Consolidation or practice as in dimension seven. Each lesson or activity has two phases: 1) Developmental phase, in which the concept is developed, and

2) Integrative phase, in which the concept is integrated into the schema or existing knowledge of the student. The developmental phase can be thought of as the awareness phase and the integrative phase as the application phase. According to Piaget, there is assimilation when new information is taken into already existing patterns and structures, and accomodation when new activities are added to an organism's repertoire or old activities are modified in response to the impact of environmental events. (Berlyne 1957) The degree of guidance will be referred to as the independent variable of the developmental phase, and consolidation or practice will be the independent variable of the integrative phase. Student interaction is part of both phases, and for purposes of this study, will remain constant in both phases of learning.

The three independent variables of student interaction, guidance in the developmental phase, and the integrative phase were chosen as important aspects of discovery teaching. Student interaction is very important according to Skemp (1971). When speaking our thoughts to another, we are also communicating them to ourselves. He also believes that audible speech is superior to sub-vocal speech in bringing ideas into consciousness more clearly and fully. "When a discussion takes place, we get this subjective effect on both sides, together with the interaction of ideas which is the more conscious purpose of those taking part." (p. 98) In addition to making ideas clearer, other students in the group can guide the individual student. (Shulman and Keislar 1966) Bruner believes in instruction providing incongruities or contrasts, as when students formulate different hypotheses. Higgins (1971) says that discovery should make great use of group processes. "One way to increase the probability of alternate solution approaches

is to increase the input of ideas fed into the problem situation. This method argues, in turn, for involving as many different sources of ideas as possible." (p. 181)

The importance of the guidance variable in the developmental phase is stated by Shulman (1970): "Those favoring learning by discovery advocate the teaching of broad principles and problem solving through minimal teacher guidance and maximal opportunity for exploration and trial-and-error on the part of the student." (p. 178) Worthen (1968) distinguishes between guided and unguided discovery. Wittrock (1966) distinguishes between guided discovery and pure discovery in his model, shown in Table 1.

TABLE 1. WITTROCK'S MODEL OF DEGREES OF GUIDANCE IN DISCOVERY (p. 187)

Rule	Solution	Type of Guidance
Given	Given	Exposition
Given	Not Given	Guided Discovery (Deductive)
Not Given	Given	Guided Discovery (Inductive)
Not Given	Not Given	Pure Discovery

The integrative phase of discovery teaching is illustrated by Bruner's position on transferability. "For Bruner, one starts with the complex and plans to learn the simple components in the content by working with the complex." (Shulman 1971, p. 188) He believes both motivation and transferability will be improved. He identifies six

subproblems in teaching a child in such a way that he will use the material appropriately. Two of these are applicable to this study. The first is practice in the skills related to the use of information and problem solving "because it turns out that however often you may set forth general ideas, unless the student has an opportunity to use them, he is not going to be very effective in their use." (p. 103) The second subproblem is discovering what it is that you have been doing and discovering it in such a way that has productive power to it. (Bruner 1966)

Descriptions of the Independent Variables

In considering each of the independent variables, the extremes will be used. Student interaction refers to the degree that students communicate with one another. Open student interaction means that students are working together and communicating with one another at least 80% of the class time. Students are working in pairs, small groups, or large group discussions. The other extreme, closed student interaction, means that the students are working together and communicating with one another for less than 20% of the class time. Open student interaction is an aspect of discovery.

Guidance in the development phase refers to the degree to which the teacher controls the content and direction of learning. Open guidance means the teacher poses a problem, lets the students pursue the problem, hypothesize, and evaluate, and controls the conversation only during the evaluation and consolidation phases of learning. Closed development means the teacher controls the entire learning sequence, either by questions to students or didactically. Open

guidance in the developmental phase is an aspect of discovery.

Concept integration phase refers to the consolidation stage of teaching. After the students or teacher have made the generalization, the practice or application of that generalization can be open or closed. Open integration means the students are presented a problem and respond to the following conditions as defined by Merrill and Wood (1974): 1) Rule finding: given labels and unencountered instances of the domain concepts and range concepts, the student is able to find or invent an operation that will complete the ordered relationship between domain and range. 2) Domain finding: given the label and unencountered instances of the range concept and the operation, the student is asked to find domain concepts with instances such that the operation will produce the instance of the range given. 3) Operation and range finding: given labels or instances of domain concepts, the student is asked to find his own operation and define his own range. 4) Domain, operation, and range finding: given a range label, the domain and operation are left to the student. The response conditions as defined here are also applicable to the open guidance of the developmental phase.

Closed integration means the student responds to the following conditions as defined by Merrill and Wood: 1) Discriminated recall, 2) Classification, 3) Rule using: given unencountered instances of domain concepts and the operation, the student is able to produce or identify the resulting instances of the range concepts.

Possible Treatments Based on Combinations of Independent Variables

An examination of the combinations of the three independent variables assigned as open or closed will lead the reader to eight combinations of the independent variables, as shown in Table 2.

TABLE 2. EXTREMES OF THE INDEPENDENT VARIABLES

Independent Variables	Closed	Open
Student Interaction	A_1	B_1
Guidance in the Developmental Phase	A_2	B_2
Integration Phase	A_3	B_3

The combinations are then:

1) A_1 A_2 A_3 No elements of discovery teaching
2) B_1 A_2 A_3 Addition of open student interaction
3) B_1 B_2 A_3 Addition of open guidance in the development
4) B_1 B_2 B_3 All elements of discovery teaching
5) A_1 B_2 A_3 Open guidance in the development
6) A_1 B_2 B_3 Addition of open integration
7) A_1 A_2 B_3 Open integration
8) B_1 A_2 B_3 Open interaction and integration

Although the student interaction could be different in the developmental phase and the integration phase, for purposes of this study, student interaction will remain constant for both phases.

Group 1 has no elements of discovery teaching. Students do not

work together, and the only student interaction that occurs is outside of class, since this cannot be controlled, and during the class discussions with the teacher as leader. The teacher controls all aspects of the learning situation. The problems assigned to students will be discriminated recall, classification, or rule using.

Group 2 introduces the open student interaction variable of discovery. Students actively work together on a problem presented by the teacher. Since the guidance of the developmental phase remains closed, there is a great deal of guidance by the teacher. The problem to be worked on consists of carefully selected examples by the teacher. The teacher also controls the direction the students are taking. In small group work the teacher monitors carefully the progress made by the group, guiding the students back on track as soon as possible if they are not going the direction intended by the teacher. Also, the integration phase consists of discriminated recall, classification, and rule using.

Group 3 introduces yet another aspect of discovery, open guidance in the developmental phase. The student interaction occurs as in Group 2. The teacher, instead of leading the students and guiding them back to the direction desired by the teacher, lets the students pursue any direction. Lessons, instead of being completed each day, could last several days. To distinguish between a daily lesson and these longer discovery lessons, the word "activity" will be introduced to refer to the longer lessons. Students in this group need to improve their evaluation skills, as most likely many different conflicting hypotheses will appear. At the end of the activity, to reinforce and consolidate the generalizations discovered, students are assigned

exercises of discriminated recall, classification, and rule using.

Group 4 has all aspects of discovery teaching. Students have open interaction and open guidance of the developmental phase as in Group 3, and in addition, at the end of the activity, students have open integration, that is, problems involving rule finding, domain finding, operation and range finding, and domain, operation, and range finding. As Ong (1976) puts it:

> In problem situations of this structure, one or two of the parameter sets are given in quite broad terms with many possible interpretations and implications embedded, and the students are required to search for a variety of solutions or solution sets for the remaining parameter sets. Flexibility is the key feature of creative problem situations. (p. 58)

Group 5 involves closed student interaction, open development, and closed integration. The teacher presents the problem to the students, and the students then work individually, making their own hypotheses and evaluating them individually. Students receive no guidance from the teacher, as in closed development, or from each other, as in open integration. Therefore, every student needs good evaluative skills to test their hypotheses. The problem with this teaching method is the possibility that students will not evaluate their hypotheses correctly and will use incorrect generalizations. Additional teacher or student guidance is needed in an individualized system to guide those students with inadequate evaluative skills. Group 5 then becomes Group 1 or Group 2.

Group 6 has the same problem as Group 5 in that students work individually on a problem without teacher guidance or student guidance. This combination is closed student interaction, open guidance in the

developmental phase, and open integration. For the same reasons as in Group 5, this is not a workable method.

Group 7 has closed student interaction, closed guidance, and open integration. Here, students work individually with teacher guidance. The only difference between this group and Group 1 is the nature of the integration. Group 7 has practice problems which are open ended. The response conditions are rule finding, domain finding, operation and range finding, and domain, operation, and range finding. This combination was the experimental treatment used by Ong (1976) and resulted in improved problem solving ability of Grade IX students in Geometry.

Group 8 involves open student interaction, closed guidance, and open integration. In this case, students are given a problem, work together to solve it with a great deal of teacher guidance through carefully selected examples and continuous help in keeping the direction desired by the teacher. Students are receiving guidance from each other and from the teacher. Also, in open integration, students are working on open-ended problems together. This is very similar to Ong's (1976) study, the difference being students working together instead of individually.

Conclusion

Three independent variables of discovery teaching were chosen for examination in this study: Student interaction, Guidance in the developmental phase, and Integration phase. Only the extremes of these variables are considered: Closed, which gives expository aspects of the variables, and Open, which gives discovery aspects. The eight

combinations of the extremes of the three variables are discussed in Groups 1 to 8. Groups 1 to 4 are possible treatment groups. Groups 5 and 6 are unworkable, while Groups 7 and 8 are similar to studies done previously. The final selection of treatments groups is discussed in Chapter IV, Design of the Study.

Conceptual Framework for Measuring Problem Solving

In measuring problem solving ability, two aspects of problem solving must be kept in mind: 1) the Process, or set of behaviors or activities that direct the search for the solution, and 2) the Product, or actual solution. (Kantowski 1977) Frequently, a correspondence between processes and products is assumed: that the individual with the largest number of correct responses possesses the best or most desirable quality of mental processes. However, this emphasis on accuracy of solutions undoubtedly gives a misleading impression of students' problem solving ability. (Bloom 1950) In problem solving, an individually acquired set of processes is used in a situation that confronts the individual. (Le Blanc 1977) In addition, problem solving strategies are largely determined by the structure of the task environment. Thus, a theory of problem solving must include a description of how different kinds of problems are solved. (Shulman and Elstein 1975) De Groot, Kleinmuntz, and Clarkson agree that knowledge of the process by which a problem is solved is at least as important as observing that it was solved.

Since processes of problem solving are important to consider, as well as varied between individuals and between problems, a study investigating problem solving should include problem solving processes.

The practical difficulty in investigating problem solving processes is finding suitable instruments to measure problem scores and to determine the processes used. According to Kantowski (1977):

> Employing a scoring scheme that considers elegance and efficiency as well as correctness of result would be desirable...A study should be undertaken to find a valid and reliable method of scoring problems so that process as well as product are considered. (p. 174)

A method of analysis of process in problem solving is to infer the mental process from the observed product. By examining the students' written solutions to a problem, it is inferred how he must have proceeded toward the solution. (Bloom 1950) The other technique used by many researchers is an interview with the subject while problem solving or immediately after.

The view of problem solving significantly influences the selection of problems and the interpretations of problem solving processes. (Lester 1977) Problem solving can be measured in many ways: 1) whether a solution is attained or not, 2) the time required to find a solution, 3) the quality of the solution, 4) the approach to the solution, as some examples. All these measures are related, but not perfectly. (Newell and Simon 1972) In constructing an instrument and devising a scoring scheme which considers process as well as product, the differing approaches to the research theories on problem solving are examined.

Approaches to Problem Solving Research

Current research in problem solving stems from several different historical streams of thought, resulting in four approaches: 1) Classical Gestalt psychology, 2) Psychometric, 3) Laboratory study

of learning, and 4) Information-processing approach. Classical Gestalt psychology deals with the analysis of internal processes, the cognitive organization, and the covariation of traits. The psychometric approach looks at the understanding of the nature of intelligence, the additive components of general mental ability, and the end products of problem solving activity. The laboratory study of learning contends that the complex processes such as problem solving obey the same laws as elementary behavioral processes of stimulus-response behavior. The information-processing approach examines the programming of computers to solve complex problems as theoretical models of human cognitive processes. (Forehand 1966)

The researcher, in examining the literature available on problem solving, decided on four scoring schemes to be used on the same set of problems to measure problem solving ability. Each scoring scheme has supporting literature focusing on different aspects of problem solving. The four scoring schemes are 1) Correct Answer, 2) Quality of Answer, 3) Polya's Four Phases of Problem Solving, and 4) Quality of Response Approaches.

Scoring Principles for Correct Answer

Hayes (1966) lends support to the scoring of problems based only on the final answer. "Some problems are solved suddenly and in a single step. Others are solved in a sequence of well-defined steps." (p. 149) Wills (1967) decided that responses were to be judged correct only if they were equivalent to the keyed responses. Being close did not count. Knifong and Holtan (1976), in analyzing written solutions to word problems, also gave points only for the correct answer. This

scoring scheme only considers the product of problem solving. The principle for this scoring scheme is to reward points only for the correct answer.

Scoring Principles for Quality of Answer

Gagne (1966) lists six internal conditions for solving a problem: 1) Recall of subordinate rules, 2) Search and selection of relevant rules, 3) Combining subordinate rules to formulate hypotheses, 4) Provisional rule selected from the hypotheses and matched to a solution model to give a general form of the answer, 5) Verification of the provisional rule by applying to a specific example, and 6) Solution rule. The individual differences which allow some students to have better answers in problem solving are the number of previously learned rules, recalling previously learned rules, distinguishing relevant and irrelevant cues, fluency in making new combinations or hypotheses, retaining a solution model, and matching a specific answer to the retained general solution model.

Kennedy, Eliot, and Krulee (1970), after examining the oral responses of twenty eight Grade XI students, list the following five steps in the act of solving algebraic word problems: 1) Reads the problem statement and forms a rough hypothesis about the kind of problem, i.e. assimilates the problem statement, 2) Looks for information requiring translation into mathematical symbols, 3) Ascertains what kind of relationships are needed to form an appropriate equation, 4) Incorporates unstated logical or physical inferences needed to solve the problem, and 5) Solves the equation.

Hollowell (1977) lists seven cognitive processes in mathematical

problem solving observed in thirty high school juniors who were asked to think aloud while attempting to solve three mathematical problems: 1) Learn or understand the problem, 2) Recall information from memory, 3) Formulate an hypothesis or general idea of how to solve the problem, 4) Attempt to find a provisional solution or develop a method of solution, 5) Check against solution model or general form of answer, 6) Verify whether the provisional solution is correct, 7) Reject the hypotheses, method of solution, or provisional solution.

S. Williams (1975) lists the following subtasks of problem solving: 1) Problem identification, 2) Hypothesizing, 3) Data collection, 4) Interpretation of data, 5) Analysis of data, and 6) Evaluation.

Weir's (1974) stages of problem solving are: 1) Statement of the problem, 2) Defining the problem by distinguishing essential features, 3) Searching for and formulating an hypothesis, 4) Verifying the solution. Weir, in listing the individual differences which affect the process of problem solving, is similar to Gagne: 1) Amount of information stored, 2) Ease of recall, 3) Concept distinction between relevant and irrelevant cues, 4) Fluency in making new hypotheses, 5) Retention of solution model, and 6) Matching instances to the general class in the verification stage. Buswell and Kersh (1956), in examining the thought processes of high school and university students as they attempted to solve six sets of problems, concluded that the failure to solve a problem is the inability to separate relevant from irrelevant facts or recognizing a need for further facts, and the inability to make a reasonable estimate.

From these models of problem solving, the researcher formulated the principles for a scoring scheme based on the quality of answers.

The principles are:

1. Students are awarded the maximum number of points if the correct answer is given, and only correct answers receive maximum points.
2. Students are awarded partial points for work which could lead to a correct solution but was incomplete or contained an error.
3. Students are awarded partial points for showing an understanding of the problem.
4. Students are awarded partial points for making correct statements about the problem but not discovering a method of obtaining the correct solution.
5. Students are awarded no points for incorrect statements about the problem or not responding to the problem.

Scoring Principles for Polya's Four Phases of Problem Solving

Polya's four phases of problem solving are: 1) Understanding the problem, 2) Devising a plan, 3) Carrying out the plan, and 4) Looking back. (Polya 1957) Many other researchers have used similar analyses of the problem solving process. Klein and Weitzenfeld (1976) list three subsections of the problem solving process: 1) Problem identification, 2) Generation of alternatives, 3) Evaluation of alternatives.

Kantowski (1977) used the following scoring scheme to find a process-product score to determine the problem solving ability of eight Grade IX Algebra students. The students earned one point for each of the following: 1) Suggesting a plan of solution, 2) Persistence, 3) Looking back, 4) Absence of structural errors, 5) Absence of

executive errors, 6) Absence of superfluous deductions, and 7) Correct result.

Troutman and Lichtenberg (1974) list the common steps in models of problem solving: 1) Must be aware of problem, 2) Must translate the problem into terms that can be handled, 3) Must generate information and strategies necessary for solving the problem, 4) Must implement strategies necessary for solving the problem, and 5) Must evaluate solutions for the problem. The specific abilities related to solving problems are: 1) Finding characteristics of objects or mathematical ideas, 2) Translating a mathematical communication into different forms, as words to numbers, graphing, diagramming, sketching, original translations, 3) Finding similarities and differences, 4) Determining sufficient, necessary, and equivalent conditions and distinguishing between essential and nonessential properties, 5) Making generalizations based on the observation of specific evidence, as finding patterns, 6) Determining alternative strategies, and 7) Approximating.

Along similar lines, Johnson and Rising (1971) list their objectives for problem solving: 1) The student identifies the question to be answered in the problem situation, 2) He selects relevant information needed to solve the problem, 3) He relates the problem to analogous situations which supply clues or solutions, 4) He draws a flow diagram of the relationships and processes involved, 5) He finds the answer to the question of the problem, 6) He generalizes the solution to other problems, 7) He applies his knowledge of mathematical concepts and processes to everyday situations, 8) He uses ideas about measurement in a construction project, 9) He uses a mathematical tool to solve a problem, 10) He identifies the role of

mathematics in a news statement, 11) He states a generalization about the pattern in a set of data.

Thorndike (1950) also lists phases of problem solving, attributed to Dewey, which are similar to Polya: 1) Becoming aware of the problem, 2) Clarifying the problem, 3) Proposing hypotheses for solution of the problem, 4) Reasoning out the implications of the hypotheses, and 5) Testing the hypotheses against experience.

Bloom (1950), in analyzing problem solving processes of college students, found the following differences in methods of problem solving: 1) Understanding of the nature of the problem, in ability to start the work on a problem, comprehension of directions, understanding the terms of the directions, and ability to solve the problem as presented, 2) Understanding of the ideas contained in the problems, in applying relevant knowledge to the problem and realizing the implication of the problem, 3) General approach to the solution of the problem, in the extent, care, and system of thought and the ability to follow through on a process of reasoning, and 4) Attitudes toward the solution of the problem, in reasoning and confidence in the ability to solve the problem.

The information-processing researchers seem to use similar processes as Polya. Paige and Simon (1966) list the steps in a flow-chart for solving word problems: 1) Input and print the problem, 2) Make mandatory substitutions, 3) Tag words by function, 4) Break into a sequence of simple sentences, 5) Transfrom simple sentences into a set of equations. In a later work, Newell and Simon (1972) list the overall organization of the problem solving process: 1) Input transformation or internal representation identifies problem solutions as obvious, obscure, or unattainable, 2) Selecting a problem solving

method, 3) Applying the selected method, 4) Termination of the method, either accepting the solution, attempting another method, selecting a different internal representation, or abandoning the attempt to solve the problem, and 5) Producing new problems or subgoals.

The scoring scheme chosen on this aspect of problem solving, which emphasizes understanding of the problem, is one devised by Taylor-Pearce (1971) based on Polya's phases of problem solving. The scoring principles for this scoring technique are:

1. The student receives points for indicating an understanding of the problem.
2. The student receives points for showing a design to solve the problem.
3. The student receives points for showing a procedure for solving the problem.
4. The student receives points for obtaining a solution.

Scoring Principles for Quality of Response Approaches

Researchers have examined the approaches students take in solving problems. Kilpatrick (1967) analyzed the solution of word problems in mathematics of fifty six Grade VIII, above-average ability, students. He identified eight process variables and reported the intercoder reliabilities as follows: 1) Deduction processes (.92), 2) Use of equations (1.00), 3) Trial and error processes (.96), 4) Reading or rephrasing (.81), 5) Checking solutions (.72), 6) Structural errors (.91), 7) Difficulty in performance (.62), and 8) Stops without solution (1.00). From these, two major production processes, deduction and trial and error, are used to identify problem solving modes of

students. He combines setting up an equation and deduction. Four groups are identified: A) Begins trial and error and uses trial and error to solve the problem, B) Between A and C, C) Infrequent use of trial and error, with a tendency to begin by another process, E) Equation or deduction. The trial and error group (group A) was more successful in solving problems than groups B or C, but group E was the most successful.

Vos (1976) found problem solving behaviors important in solving mathematical problems: 1) Drawing a diagram, 2) Approximating and verifying, 3) Constructing an algebraic equation, 4) Classifying data, 5) Constructing a chart, 6) Recalling a formula, 7) Searching for a pattern, 8) Constructing a physical model, 9) Asking a missing question, and 10) Trying to solve a related but simpler problem.

Travers et al. (1977) lists twelve approaches to problem solving: 1) Select appropriate notation, 2) Make a drawing, figure, or graph, 3) Identify wanted, given, and needed information, 4) Restate the problem, 5) Write an open sentence, 6) Draw from a cognitive background, 7) Construct a table, 8) Guess and check, 9) Systematize, 10) Make a simpler problem, 11) Construct a physical model, and 12) Work backwards.

Clement (1977) interviewed third and fourth grade students as they attempted to solve quantitative story problems. The approaches they used were: 1) Acted-out solutions, 2) Counting-based solutions, 3) Solutions via a number sentence, 4) Immediate solutions, 5) Solutions via written symbolic algorithms, 6) Use of diagrams, 7) Spontaneous activities related to inverse, commutative, associative, and distributive principles, 8) Solving a problem in pieces, 9) Using

more than one approach, 10) Generating related problems, and 11) Convergent trial and error approaches.

Higgins (1971) lists three patterns of problem solving: 1) Guessing an answer, working out the consequences, comparing the result with the original conditions of the problem, and improving the original guess, 2) Finding a simpler related problem or part of the original problem by ignoring certain conditions, 3) Decomposing and recombining, focusing on details individually.

Lefrancois (1972) lists four decision sequences on concept learning which also seem to apply to problem solving: 1) Simultaneous scanning, generating all possible hypotheses and using each successive selection to eliminate all untenable hypotheses, 2) Successive scanning, or trial and error, 3) Conservative focusing, accepting the first positive instance as the complete hypothesis, then check and refine choices, and 4) Focus gambling, varying more than one value at a time.

Boychuk (1974) developed a scoring scheme based of efficiency of the problem response. After listing the possible solutions and approaches, they were ranked according to which method resulted in the solution in the easiest and quickest way. The most straightforward approaches received a score of two, other correct approaches received a score of one, and incorrect approaches were scored zero.

In considering a scoring scheme focusing on the approaches to problem solving, selected elements from these studies were considered in the scoring principles of the quality of response approaches:

1. Response approaches are ranked according to the feasibility to lead to a correct solution and mathematical competence.

2. Approaches do not have to be carried out to a complete solution to be given top ranking.
3. No response gets bottom ranking.
4. Next ranking is a guess without verification.
5. Restating the problem gets next ranking, including a drawing.
6. Guess and verify gets middle ranking.
7. Substituting the information into a known formula gets next ranking.
8. Using a table or pattern has next to the top ranking.
9. Deductive reasoning and making an equation gets top ranking. Different equations, depending on completeness and accuracy, get different rankings.

Conclusion

The principles for each of the four scoring schemes have been developed from the literature of problem solving. In Chapter V, Treatments and Instrumentation, the principles are developed into the four scoring schemes of Correct Answer, Quality of Answer, Polya's Four Phases of Problem Solving, and Quality of Response Approaches.

Chapter IV

DESIGN OF THE STUDY

Introduction

The developmental phase of the study involves the development of treatments for a unit in Mathematics 10, Rational Expressions. In addition, an achievement test for the unit and a problem solving test related to the unit were developed by the researcher for use in the experimental phase of the project. The treatments are described in detail in the Appendix, and the instruments are discussed in Chapter V.

The experimental phase involves the application of the treatments and administration of the tests. The purpose of this chapter is to specify the selection of treatments, the hypotheses, the population and sample, the method of data collection, and the analysis of the data.

Selection of Treatments for Study

Ideally, all eight combinations of the three independent variables resulting in the eight groups described in the Conceptual Framework would be tested. However, for time factors and managability purposes, only three groups will be studied. Groups 5 and 6 do not seem managable as explained, and Group 7 has been tested by Ong. Group 8 is a variety of Ong's using open student interaction, and would be an interesting group to study if not for managability considerations. The first four groups make a progression of adding one discovery variable to each treatment, starting with no discovery variables and ending with all discovery

variables. However, since only three groups could be chosen for the study, one of the first four groups had to be omitted. For greatest contrast, Group 1, with no discovery variables, and Group 4, with all discovery variables, were chosen as part of the study. Of the remaining two groups, Group 3 has two discovery variables, open interaction and open guidance, while Group 2 has one discovery variable, open interaction. Since the researcher is interested in discovery teaching, the group with the most aspects of discovery teaching, Group 3, was chosen. Hereafter in the study, Group 1 will be referred to as E (Expository), Group 3 as D1 (Discovery I), and Group 4 as D2 (Discovery II).

Hypotheses

Question 1. Are students able to distinguish between expository and discovery treatments developed on the same unit topic of a mathematics course?

1A. There is a significant difference between the treatments of Group E and Group D1.

1B. There is a significant difference between the treatments of Group E and Group D2.

1C. There is a significant difference between the treatments of Group D1 and Group D2.

Question 2. Are there differences in achievement between students taught by a treatment involving no aspects of discovery teaching (E) and treatments involving aspects of discovery teaching (D1 and D2)?

2A. There is no significant difference in the achievement scores of students in Group E and Group D1.

2B. There is no significant difference in the achievement scores of students in Group E and Group D2.

2C. There is no significant difference in the achievement scores of students in Group D1 and Group D2.

Question 3. Are there differences between students taught by an expository treatment (E) and discovery treatments (D1 and D2) in problem solving ability as measured by four different criterion variables?

3A. There is a significant difference in the problem solving ability of students in Group E and Group D1.

3B. There is a significant difference in the problem solving ability of students in Group E and Group D2.

3C. There is a significant difference in the problem solving ability of students in Group D1 and Group D2.

The three hypotheses above were tested separately for each of the four criterion variables: 1) Correct Answer, 2) Quality of Answer, 3) Polya's Four Phases of Problem Solving, and 4) Quality of Response Approaches.

Population and Sample

The subjects used in this research study were Grade X, Mathematics 10, students who are attending Edmonton Public Schools. Three classes from the same high school were involved in the experiment. Twelve volunteers from each class were selected by the classroom teachers. The teachers were asked to divide the class into high, medium, and low achievement groups and select four volunteers from each achievement group. On the basis of their most recent mathematics grade, a computation for the analysis of variance resulted in an F-ratio of 1.47.

Since the value of F required for significance at the .05 level is 3.32, the conclusion is no significant difference between the groups on the basis of mathematics achievement.

Sample Attrition

The thirty six students remained in the treatments eleven days. In Group E, one student was absent four days, another two days, and four students one day each. One day had three students absent, another day had two students absent, and three days had one student absent. These absences were not significant in the effect of the treatment.

In Group D1, one student was absent four days, three students were absent two days, and two students were absent one day each. One day had three students absent, two days had two students absent, and five days had one student absent. These absences were not significant in the effect of the treatment.

In Group D2, one student was absent three days, one student was absent two days, and three students were absent one day each. One day had three absences, one day had two absences, and three days had one absence. These absences were not significant in the effect of the treatment.

Since all students remained in the treatments the eleven days, with absences as reported not significant, the sample size for Hypothesis 1 was thirty six.

Two students were absent when the achievement test was given. One student was from the lower achievement students of Group E and the other student was from the lower achievement students of Group D1. The Sample size for Hypotheses 2 and 3 was thirty four.

Administration of Treatments

The researcher worked with all three groups of twelve students. Recognizing researcher bias, this procedure seemed the most managable, in that a teacher or teachers would not need to be trained in the three treatments. A smaller group of twelve rather than the entire class was felt to reduce discipline as a factor of the research treatments. A cross-section of twelve students from one class still gives the variety of ideas and roles needed in a discovery lesson.

The unit lasted thirteen days, the same number of days that the regular teachers spent on the unit. Since two days were needed for testing, the instruction time was eleven days, eighty minutes per day. All three treatments began on the same day. The same topics were covered in the three treatments, with the order differing between Group E and Groups D1 and D2, as explained in detail in Chapter V.

Choice of Treatment Topic

The unit on rational expressions was chosen as the topic for the treatments. It is one of the required units of study in the Mathematics 10 curriculum. Since the researcher took twelve students from each of the cooperating classes, it was felt that the same topic as the rest of the class should be used. Also, the researcher wanted to test the claim that any unit of mathematics study is appropriate for a discovery approach. The researcher is also interested in the adaptation of one unit of study to three treatments, especially with a topic which is part of the regular curriculum.

Collection of the Data

The instrument used to collect data for Hypothesis 1 is the Student Inventory of Teacher Behavior, developed by Naciuk (1968). The researcher and one of the regular classroom teachers developed the achievement test, Achievement in Rational Expressions, to test Hypothesis 2. The researcher developed the problem solving test, Problem Solving in Rational Expressions, based on the content of the unit, to test Hypothesis 3. Four scoring schemes, Correct Answer, Quality of Answer, Polya's Four Phases of Problem Solving, and Quality of Response Approaches, were used to collect data on problem solving ability. The test items, in combination with each scoring scheme, yields four test of problem solving ability. In addition, attitude interviews on the treatments were conducted by the researcher with the students in Groups D1 and D2.

Following the eleven days of treatment, the Student Inventory of Teacher Behavior was administered on the twelfth day of the experiment. The students were given ten minutes to complete the inventory. The Problem Solving in Rational Expressions Test was administered in the remaining seventy minutes of the class period. The students were given the entire eighty minutes of the thirteenth day for the Achievement in Rational Expressions Test. The attitude interviews were done on a one-to-one basis with the researcher following the last day of testing. Students were taken from the regular classroom for approximately three minutes to complete the interview.

Analysis of the Data

Hypothesis 1. The Student Inventory of Teacher Behavior was used to determine whether there is a significant difference between the three treatments. This is especially important since all three treatments were done by the same teacher. To determine the difference, the Significance of the Difference Between Two Means was found.

Hypothesis 2. The computation for the Analysis of Variance, One-way Classification, was used to determine the difference between the three groups on the Achievement in Rational Expressions Test.

Hypothesis 3. The Problem Solving in Rational Expressions Test was analyzed with the Analysis of Variance, One-way Classification. Each scoring scheme was analyzed separately. The calculation of the correlation coefficient from ungrouped data using deviation scores was used between each two scoring schemes of the Problem Solving in Rational Expressions Test.

Delimitations

The study was delimited in the choice of treatments. Only three of the eight independent variables of discovery teaching were examined. Also, only the extremes of the variables appear in the treatments. Three of the eight possible combinations of independent variables were treated.

The study was delimited in the choice of subjects. The subjects were Mathematics 10 students, generally the brighter students in Grade X. One school in the Edmonton Public Schools was used. Instead of the entire class of approximately thirty students, a cross-section of twelve

students were used as the treatment group for each class.

The study was delimited in length of treatment time. One unit of study, Rational Expressions, lasting eleven days of instruction time and thirteen days total, comprised the treatment.

The study was delimited in the examination of problem solving ability. Inferences were made about the students' processes of problem solving from their written solutions, rather than an interview during or after the act of problem solving. Problems were chosen to reflect the subject matter covered in the unit rather than problem solving in other areas. Four scoring schemes for problem solving were developed.

Chapter V

TREATMENTS AND INSTRUMENTATION

Treatments

The researcher taught three groups of twelve students each for eleven class periods of eighty minutes in length. The number of class periods spent on the unit was determined by the regular classroom teachers.

The title of the unit is Rational Expressions. The topics covered within the unit are definition of a rational expression, permissible values, simplifying rational expressions, multiplication, division, addition, and subtraction of rational expressions, division of polynomials, dividing by a monomial, dividing by a polynomial, rational expressions in open sentences, word problems, and complex rational expressions. Group E covered the topics in the order given.

Group D1 covered the following topics in the order given: rational equalities, rational equalities with quadratics, rational inequalities, word problems, complex expressions, and dividing by a polynomial. Each topic was followed by practice exercises of the same kind as in the activity.

Group D2 covered the following topics in the order given: rational equalities, rational inequalities, word problems, complex expressions, and dividing by a polynomial. Rational equalities was followed by problems involving quadratics, as was the topic rational inequalities. The problems for each topic led to new discoveries, rather than being the same type as in the activity.

Group E is based on Group 1 of the conceptual framework. The teacher introduced the lesson, then lead the class through various examples of the object of the lesson. The students were then assigned practice examples from their textbook to solve for the remainder of the eighty minute period. The lesson was concluded each day.

Group D1 is based on Group 3 of the conceptual framework. The students were given a list of problems to solve. The teacher provided no guidance while the students attempted to solve the problems. The students worked in pairs to solve the problems, then shared their solution method with the rest of the group. The teacher acted as recorder only as the students stated their hypotheses. At the conclusion of the recording of all generated hypotheses, an evaluation by the students of the hypotheses occured, with the teacher providing guidance at this stage. A consolidation period followed, with exercises given which are like the problems the students solved to make their hypotheses. The activities lasted as many days as needed by the students.

The complete lessons appear in the Appendix. An example, from Activity 1, begins with a list of nineteen problems. Two of these were: 2. $3/x+5/x=4$ and 14. $4/x+2/(x+3)=1/(x^2+3x)$. One of the hypotheses which followed was: To solve equations involving rational expressions, a) Cross-multiply, b) Guess, c) Multiply by a common denominator, d) Combine fractions on one side, then cross-multiply. Two examples of the practice exercises were: 4. $(a-1)/3+(a+2)/6=2$ and 8. $1/4a^2+5/6a=2/3a^2$.

Group D2 is based on Group 4 of the conceptual framework. The student activites are as given in Group D1. The difference is in the

integration phase. Group D2 is given problems which lead to new hypotheses. For example, the same two problems from the list of nineteen from Activity 1 lead Group D2 to a similar hypothesis as Group D1: To solve equations involving rational expressions, a) multiply by the common denominator, b) cross-multiply, c) combine terms, then cross-multiply. Integration problems were like 1. $2/(n-3)+2/n=1$, which contained a squared term when simplified. The students then hypothesized about these problems: 1. To solve equations with a square term, get all members on one side and set the side equal to zero. 2. There are two answers when there is a square term. 3. Check both answers, as one may give a zero denominator. The students from Group D2 never practiced exercises like the problems they used to make the hypotheses.

Unlike the Group E lessons, which were introduced and completed within the same eighty-minute period, the activities of Groups D1 and D2 usually took more than one period. The beginning and conclusion of activities was determined by the progress of the students, not by the time constraint of the eighty-minute period.

The order of presentation of topics is different for Group E and Groups D1 and D2. Groups D1 and D2 began with a need for simplifying, adding, subtracting, multiplying, and dividing rational expressions through solving equations involving rational expressions. Group E started with the basic ideas about rational expressions which became more complex, but a need for the skills of manipulating rational expressions was not apparent until the seventh lesson. Groups D1 and D2 learned the subordinate skills of manipulating rational expressions by using them to solve equations. The remaining lessons and activities

followed the same order.

Instrumentation: Design, Validity, Reliability

1. Student Inventory of Teacher Behavior (SITB)

Design. The SITB, whcih appears in the Appendix, was taken from a study by Naciuk (1968). He developed the instrument "to assess the extent to which the teachers were able to follow the expository or mathematizing models in their teaching." (p. 50) The SITB scores represent general impressions of classroom activities and behavior. All students rated the teacher on thirty items using a five-point scale of Almost Always, Often, Sometimes, Seldom, and Almost Never. The items were worded for the expository and mathematizing methods, so the five-point scale read from left to right for the expository wording and right to left for the mathematizing wording. In this way, the choices to the extreme left indicated the ideal expository method and those to the extreme right indicated the ideal mathematizing method. The weighting factors for each item were then always in the order 0, 1, 2, 3, 4.

Scoring. The students' responses were counted for each item, and the percentages of the students answering the items each way were calculated. The percentages were then multiplied by the weighting factors. The weighted percentages were added and the sum was divided by four. This quotient is called the class consensus rating for the item, with an ideal expository score of zero and an ideal mathematizing score of one hundred. To illustrate, item one from Group E is shown in Table 3.

TABLE 3. ILLUSTRATION OF THE SCORING OF THE STUDENT INVENTORY OF TEACHER BEHAVIOR

1. When an answer is wrong, our teacher ___ tells us immediately.

A. Almost Always B. Often C. Sometimes D. Seldom E. Almost Never

Twelve students responded to the item in Group E.

	Almost Always	Often	Sometimes	Seldom	Almost Never
Frequency	4	6	2	0	0
Percent	33.33	50	16.67	0	0
Weight	0	1	2	3	4
Product	0	50	33.33	0	0
Sum	83.33				
Class Consensus Rating		$83.33 \div 4 = 20.83$			

There are six categories for the items: Teacher Omniscience (Items 1-5), Introduction of Generalization (Items 6-10), Control of Pupil Interaction (Items 11-15), Method of Answering Questions (Items 16-20), Uses of Student Responses (Items 21-25), and Method of Eliminating Falso Concepts (Items 26-30). The rating of the teacher's behavior for each category was obtained by finding the mean of the class consensus ratings of each of the five items in the category.

Interpretation of Categories. Naciuk (1968) describes each of the categories of items for the expository model, which is similar to Group E in this study, and the mathematizing model, which is similar to Groups D1 and D2.

1. Teacher Omniscience: In the expository model, the teacher is the authority on the mathematics under discussion. The teacher demonstrates appropriate procedures for solving problems, directly helps students solve problems, and corrects student mistakes immediately. The mathematizing teacher presents problems but allows the students to find methods of solution. Students determine the correctness from their knowledge of mathematics, rather than the teacher pointing out errors.

2. Introduction of Generalization: The expository teacher presents a rule for solving problems, presents examples, and states a generalization of the rule. In the mathematizing model, students generate hypotheses after attempts at problem solutions. The generalizations are formulated from the hypotheses after class discussion and evaluation of hypotheses presented.

3. Control of Pupil Interaction: The expository class focuses on the correct method of solving a problem. The method is broken into certain steps based on the rules of the problem. The teacher directs students to this precise solution of the problem. The mathematizing class pursues a variety of methods of solution to a problem. After exploration of the problem, which involves no control of pupil interaction, the teacher guides the students to generalizations.

4. Method of Answering Questions: The expository teacher answers questions directly, either by restatement of the rule or generalization which applies to the problem or solving the problem for the student. In the mathematizing model, the teacher is an advisor, only rewording or regrouping examples when students have questions.

5. Use of Student Responses: In the expository model, the teacher

uses student responses as feedback on understanding of the lesson. However, in a mathematizing lesson, the student responses are the main part of the lesson. The teacher does not evaluate the responses, rather the class evaluates responses.

6. Method of Eliminating False Concepts: The expository teacher warns students of common errors. Incorrect responses are immediately corrected. Students are given practice examples that do not overgeneralize a concept. In the mathematizing model, the students are given problems deliberately to overgeneralize the rule. Also, students try a variety of methods to solve problems. The elimination of false concepts comes from the students and their knowledge of the mathematics in the evaluation stage at the end of the lesson.

Relationship of SITB to Treatments of Groups E, D1, D2. The six categories of Naciuk's SITB correspond to the three aspects of teaching in this study. Categories 1, 2, 4, and 5 correspond with the guidance in the developmental phase, Category 3 corresponds with the student interaction, and Category 6 corresponds with the integration phase. The Group E treatment corresponds to the expository model descriptions in each of the six categories of the SITB. Group D1 treatment corresponds to the mathematizing model in Categories 1 to 5 and the expository model in Category 6. In particular, Item 27 of the STIB should have an expository rating for Group D1: In our assignments we are ___ given problems which cannot be solved using rules discussed in class.
A. Almost Never B. Seldom C. Sometimes D. Often E. Almost Always.
The other items of Category 6 should have a mathematizing rating.
Group D2 treatment corresponds to the mathematizing model in all six categories. Therefore, in the ideal, Group E would have a class

consensus rating of 0 for all items, while Group D1 would have a class consensus rating of 100 for all items except 27, which would have a rating of 0, and Group D2 would have a class consensus rating of 100 for all items.

Validity. The researcher has extablished the relationship between the SITB and the treatments of the research project. The validity of the instrument was established by Naciuk (1968) in four ways: 1) face validity, since the definitions of each category describes the things teachers do in the mathematics classroom, 2) inclusiveness of categories to cover the total activities of the lesson, 3) content validity from seven qualified judges experienced in expository and mathematizing teaching methods, and 4) pilot study results showing discrimination between expository and mathematizing teachers on all items.

Reliability. Naciuk (1968) established reliability through the test-retest method in the pilot study. The results yielded a Spearman r_s of .75 for the mathematizing class and an r_s of .94 for the expository class, which is significant at the .05 level. While the r_s of .75 was not significant at .05, Naciuk argued that it was sufficiently high to indicate a degree of reliability in light of variation in teaching method and a three week difference in testing dates.

2. Achievement in Rational Expressions Test (ARET)

Design. The ARET of forty items (See Appendix) was designed by the researcher and the regular teacher of Groups D1 and D2. The items were of the type of exercises found in the textbook on the unit studied,

"Chapter 4. The Algebra of Rational Expressions." (Nichols, 1970) The number of items from each section of the unit was determined by a consensus of the regular teacher and the researcher, an attempt being made to keep the number of questions from each section equal. Each question assigned to the students in class work had a representative question on the ARET.

Validity. Since the test items were selected as representative of the questions in the textbook, the ARET has content validity. Table 4 shows the percentage of the test which covers each lesson or activity.

TABLE 4. PERCENTAGES OF THE TEST COVERING EACH LESSON OR ACTIVITY

Group E										
Lesson	1	2	3	4	5	6	7	8	9	10
Percentage	6	13	9	11	11	6	16	9	8	11
Group D1										
Activity	1	2	3	4	5	6				
Percentage	38	39	9	8	3	3				
Group D2										
Activity	1	2	3	4	5					
Percentage	77	9	8	3	3					

In Group E, the items are closely representative, in type and proportion, of the content presented in the unit. In Groups D1 and D2, the first activities lasted most of the experimental time, so the items are

representative, in type and proportion, of the content presented in the course.

Reliability. The split-half method of estimating reliability was used. The test items were divided into two halves by odd and even numbers of the items. The two resulting scores were correlated at .904. The Spearman-Brown extimate of reliability of the whole test was .947. From these results, the ARET was judged to be internally consistant.

3. Problem Solving in Rational Expressions Test (PSRET)

Design. To test the effect of the aspects of discovery teaching on problem solving ability, a paper and pencil test of eight items was constructed by the researcher. The items were selected as pertaining to the unit of study, Rational Expressions. All of the items involve techniques of solution which were not covered in the unit, thereby making them problems, not merely exercises to which a rule is memorized and applied. The problems have one answer, thereby involving convergent thinking. The students were given instructions to write everything about the problems they could think of, then to attempt the solution as far as possible. The test items appear in the Appendix.

Correct Answers Scoring Scheme. The first method of scoring the PSRET considered only the solution. Students who gave the correct answer received one point, and those who gave an incomplete answer or an incorrect answer received no points. This scoring scheme emphasizes the purpose of problem solving, to find a correct solution.

Quality of Answers Scoring Scheme. In scoring the PSRET according to the quality of answers, each item was assigned a maximum number of points. Items 1, 4, 7, and 8 were assigned ten points each, and items

items 2, 3, 5, and 6 were assigned fifteen points each. Students were awarded the maximum points if the correct answer was given, and only correct answers were given the maximum points. Students were awarded partial points for work which could lead to a correct solution but was incomplete or contained an error. Students also earned partial points by making correct statements about the problem but not discovering a mehtod of obtaining the correct solution. Students were awarded no points for incorrect statements about the problem or no response. The emphasis is on obtaining the correct solution and steps taken to find the solution. The scoring scheme appears in Table 5.

TABLE 5. QUALITY OF ANSWERS SCORING SCHEME

Points Awarded	Reason for Points Being Awarded
10	Any correct answer
9	Would-be correct answer, but a mistake in computation
8	Correct equation or pattern, but no or wrong answer
7	One correct answer or partial answer which can be found somewhere in the students work
6	Correct explanation of the problem
5	Starting correctly, but making an incorrect assumption
4	Attempting a solution using more than two correct steps. Guess which comes close to the answer which indicates understanding of the problem.
3	Any equation or pattern with a correct statement
2	Any correct statement about the problem
1	Any response
0	No response

Items 2, 3, 5, and 6 were awarded fifteen points for correct answers and partial points corresponding to the categories given in the table.

Polya's Four Phases of Problem Solving Scoring Scheme. Taylor-Pearce (1971) devised a marking scheme, see Table 6, for problem solving based on Polya's four phases of problem solving: 1) Understanding the problem, 2) Devising a plan, 3) Carrying out the plan, and 4) Looking back. From these broad areas, he made ten yes or no questions for the scorer to ask about each item of the test. Equal weights were given to each yes, worth one point, and each no, worth zero points. The students score for each item was the sum of the ten component scores.

TABLE 6. POLYA'S FOUR PHASES OF PROBLEM SOLVING SCORING SCHEME

Understanding the Problem

1. Did the student indicate expressly or implicitly that he had at least a partial understanding of the problem? yes/no

2. Did the student indicate expressly or implicitly that he had a complete understanding of the problem? yes/no

Design

3. Did the student give evidence expressly or implicitly that he had a design to solve the problem? yes/no

4. Was the design such as would possibly lead to a complete solution? yes/no

Procedure

5. Did the student show some mathematical competence in the pursuit of his design? yes/no

6. Did he discover significant relationships which could effectively lead to a solution of the problem? yes/no

7. Did he effectively use these relationships to obtain a solution? yes/no

Solution

8. Did the student obtain a partial solution? yes/no

9. Did he indicate that a complete solution exists? yes/no

10. Did he obtain a mathematically complete solution? yes/no

Quality of Response Approaches Scoring Scheme. In order to score the problem solving test using this method, all of the response approaches for each item were recorded without student name or treatment group. The different approaches were tallied. One item had five different response approaches, while another item had over twenty. The approaches were then ranked according to feasibility to lead to a correct solution and mathematical competence. To determine the ranking of the approaches, the guidelines in Table 7 were used. The student did not need to complete the approach to a correct solution to gain top ranking. The top ranking received a score of one hundred percent and the bottom ranking received a score of zero percent. The rankings between were percentages of the ratio of the ranking to the highest ranking. For instance, Item 1 had eleven approaches, with the rankings ranging from zero to ten. A student with a ranking of three then received a score of 3/10, or 30 percent. This method of scoring the PSRET emphasized the approach students took to finding the solution to the problem, or the process to the problem.

The categories given in Table 7 are general categories for approaches. There may be different approaches within the same category,

especially Category 7. Each distinct equation was considered a different approach. The ranking within the category depended on completeness and accuracy of the equation. The categories of responses for each problem appear in the Appendix.

TABLE 7. QUALITY OF RESPONSE APPROACHES SCORING SCHEME

Category	Description of Category
1	No response
2	Guess without verification
3	Restating the problem (includes drawings)
4	Guess and verify (trial and error)
5	Substitute information into known formula
6	Use of a table or pattern
7	Deductive reasoning or use of an equation

Validity. Since the definition of problem involves presenting a challenge that cannot be resolved by some routine procedure known to the student, the test items were chosen such that they did not apply directly to an algorithm taught in the unit. The researcher found solutions to the test items to ensure that they were problems according to the definition. Items 1 and 2 involve the separation of a rational expression, which uses different algebraic steps from adding or subtracting an algebraic expression, for which algorithms were taught. Items 3 and 4 involve rates. Although the students worked with questions which could be solved using the distance formula ($d = rt$), the

items of the PSRET involve more than one application of the distance formula. Items 5 and 6 are work problems with multiple applications of the algorithm for work problems needed to solve them. The recognition of place value applications and the ability to translate word sentences into mathematical sentences are two useful techniques for solving Item 7. Item 8 involves the recognition of number patterns. Upon examination of the activities required to answer the items, the items can be classified as problems. The PSRET therefore has face validity in that the students are asked to solve items that are problems by definition.

The PSRET is not intended to be an achievement test, in which the students are tested for direct application of algorithms taught or discovered in the treatments. A high correlation between achievement scores and problem solving scores would indicate less construct validity since the PSRET is intended to differ from the ARET. (Campbell 1959) However, high achievers would be expected to perform better on a problem solving test than low achievers. (Thorndike 1922, Kinsella 1970, Balow 1964, Martin 1963) Therefore, there should be a positive correlation between ARET scores and PSRET scores. In using the construct validation procedure of group differences, Cronbach (1955) states:

> Only coarse correspondence between test and group designation is expected. Too great a correspondence between the two would indicate that the test is to some degree invalid, because members of the groups are expected to overlap on the test. (p. 287)

Hence, the correlations of .618, .639, .738, and .475 between the ARET and the four scores of the PSRET support the construct validity of the PSRET.

Having established that the items are problems and that the items are different from the achievement items, the next step is to examine the processes used by the students in solving the problems. This will involve examination of the scoring schemes. Helmstader (1970) included an analysis of scoring procedures as a method of establishing construct validity. Cronbach and Meehl (1955) also refer to studies of process and scoring procedures as a method of establishing construct validity.

The first scoring scheme, Correct Answer, awards points only for correct answers. Since the items have been established as problems, those who arrive at the solution are successful problem solvers, while the other students are not. Since successful problem solvers are awarded, construct validity is supported.

Gagne (1966) lists the internal processes necessary to problem solving as: 1) Recall of previously learned rules and concepts, 2) Search and selection of the recalled rules which are relevant to the problem, 3) Combining the previously learned rules and concepts, 4) Arriving at a provisional rule believed to solve the problem, 5) Verification of the rule by carrying out the operations suggested, and 6) Arriving at the solution. The Quality of Answers method of scoring applies to Gagne's problem solving steps. Awarding 1 and 2 points for any response or any correct statement about the problem indicates recall of previously learned rules and concepts. Three points are awarded for a correct statement in an equation or pattern, which indicates a selection of a relevant rule. Awarding 4, 5, or 6 points for attempting solutions or guessing which indicates understanding, or correctly explaining the problem indicates combining the previously learned rules. Seven and eight points for a correct partial

answer and a correct equation show arriving at a rule believed to solve the problem. Nine points awarded for a would-be correct answer with a mistake in computation indicates verification of the rule by carrying out the operations. Finally, ten points for the correct answer indicates that the student has carried out all of the previous processes correctly. This matching of scoring scheme to problem solving steps of Gagne supports construct validity for the Quality of Answers method of scoring problem solving.

Polya's Four Phases of Problem Solving method of scoring have already been shown to correspond with Polya's problem solving processes, which are somewhat different from Gagne's problem solving processes. Therefore, construct validity is supported by this scoring procedure.

Skinner, in his operant analysis of problem solving, is interpreted to be interpreted by be referring to response approaches to problem solving rather than the activities within the solution. (Skinner 1966) By evaluating the response approaches in the students' solutions to the problems, the researcher was again supporting the construct validity of the problem solving test.

The above discussion provides evidence for supporting validity for the PSRET items and scoring techniques.

Reliability. Kantowski (1977) recommends intercoder reliability in a test of problem solving ability in which processes in problem solving are investigated as well as product. In addition, the researcher used the split-half method of estimating the reliability for each scoring scheme.

The Correct Answer scoring scheme only considers the product of the students' effort in problem solving. To use interrater reliability

with this scoring scheme seemed trivial to the researcher; therefore, interrater reliability was not used. The results of the Spearman-Brown estimate of reliability for split-half testing, using odd and even numbered problems for the two halves, was .637. Since the problems were conceptually different and there were only eight problems, the split-half estimate does indicate some reliability.

The process of estimating test reliability by correlating two sets of scores was used on the Quality of Anwers scoring scheme. Three graduate students and two professors in Mathematics Education scored a random sample of ten tests, four items from the eight-item test. First, each of three raters scored a different sample using the ten-point scoring scheme divised by the researcher. The scores were correlated by item with the researcher's scores. As shown in Table 8, Items 1, 3, 4, and 6 had low reliability estimates. The researcher interviewed the raters to determine difficulties with the criteria of the scoring scheme, then made appropriate revisions in the wording to agree with the researcher's conception of the scoring scheme. The revised wording of the ten-point scoring scheme was then used by Raters 4 and 5 on the same samples as Raters 1 and 2. Item 6 still had a low reliability estimate of .345, less than the critical value of .632 for significance at .05. This can be explained somewhat by the low scores given due to the difficulty of the problem. Also, the raters were not trained by the researcher, only given the scoring scheme and asked to use it. The researcher chose to accept the reliability estimates as reported.

A split-half reliability on odd-even items was also used for the Quality of Answers scoring scheme. The Spearman-Brown estimate was .780 and accepted by the researcher.

TABLE 8. INTERRATER RELIABILITY FOR QUALITY OF ANSWERS

Item	Rater 1	Rater 2	Rater 3	Rater 4 (1)	Rater 5 (2)
1	.994		.457	.991	
2	.764		.640	.805	
3	.337		.962	.629	
4	.479		.925	.676	
5		.793			.890
6		.324			.345
7		.997			.973
8		.784			.977

For an estimate of reliability with Polya's Four Phases of Problem Solving, one other rater besides the researcher used the ten-point scale developed by Taylor-Pearce (1968). The second rater, a graduate student in Elementary Curriculum Studies, with mathematics experience, scored a sample of ten papers on all eight test items, given only the scoring scheme and test papers. As shown in Table 9, only Item 4 was below the critical value of .632 of the correlation coefficient. The researcher accepted the reliability.

The split-half reliability on odd-even items for Polya's Four Phases of Problem Solving, Spearman-Brown estimate, was .816 and accepted by the researcher.

TABLE 9. INTERRATER RELIABILITY OF POLYA'S FOUR PHASES OF PROBLEM SOLVING

Item	1	2	3	4	5	6	7	8
Reliability Estimates	.870	.962	.693	.425	.953	.915	.834	.833

The Quality of Response Approaches scoring scheme required a multiple-rater estimate of reliability. Six raters from Mathematics Education, four graduate students and two professors, and the researcher scored the same samples of ten tests on each test item.

Each rater used a seven-point scale, Tables 10 to 12, of categories of approaches constructed by the researcher to provide a simplified version of the ranking of responses by the researcher. Three seven-point scales were used, one for Items 1 and 2, one for Items 3, 4, 5, and 6, and one for Item 8. The simplified scoring schemes were used in the reliability estimates rather than the researcher's scoring schemes to simplify conditions for the raters. The raters were given the categories and sample items without training.

The method of estimating the reliability of ratings is based upon the analysis of variance. (Ebel 1951) The formulae yield the reliability of average ratings and the reliability of individual ratings. Both estimates of reliability are reported in Table 13, but since the raters worked individually, and in practice only one score would be obtained, the estimate of reliability of individual ratings should be considered primarily.

TABLE 10. QUALITY OF RESPONSE APPROACHES RELIABILITY CATEGORIES 1 AND 2

Category	Description of Category
1	No response
2	Guess without verification
3	Restating the problem
4	Guess and verify (trial and error)
5	Separate the numerator, keep the denominator as given
6	Division into equal parts (ie. divide by 2 or 3)
7	Deductive steps to separate the numerator and denominator

TABLE 11. QUALITY OF RESPONSE APPROACHES RELIABILITY CATEGORIES 3 TO 7

Category	Description of Category
1	No response
2	Guess without verification
3	Restating the problem
4	Guess and verify (trial and error)
5	Substitute information into known formula (eg. rate)
6	Table or pattern
7	Making an equation (deductive reasoning)

TABLE 12. QUALITY OF RESPONSE APPROACHES RELIABILITY CATEGORIES 8

Category	Description of Category
1	No response
2	Guess without verification
3	Computation in one part, no response in other part
4	Computation in both parts, no use of a pattern
5	No response first part, use of pattern in second part
6	Computation in first part, use of pattern in second part
7	Use of pattern in both parts

TABLE 13. INTERRATER RELIABILITY OF QUALITY OF RESPONSE APPROACHES

Item	Reliability of Individual Ratings	Confidence Limits (5%) for Individual Ratings		Reliability of Average Ratings
		Upper 5%	Lower 5%	
1	.225	.427	.175	.670
2	.573	.743	.367	.904
3	.408	.610	.208	.828
4	.679	.818	.490	.937
5	.588	.755	.384	.909
6	.445	.651	.227	.828
7	.360	.566	.167	.797
8	.811	.908	.648	.968

Items 1, 3, 6, and 7 have low reliability estimates, yet the upper confidence limits of all but Item 1 are above .500. The lower confidence limits of all but Item 8, however, are below .500, but are all positive. Difficulties among the raters with Category 7, Making an equation, contributed to the low reliability. Since the categories were hierarchical, many students who made equations had actually poorer approaches than other students using a different approach. The researcher had many levels of approaches within Category 7 to take this into consideration, but the simplified version of the rating scale that the other raters used allowed for no differences. Difficulty with Item 1 seems to lie with Categories 2, 5, and 7. Raters differed in their interpretation of guessing and deductive steps.

Eventhough the reliability estimates were low, the researcher accepted the results in consideration of the previous discussion and the fact that the raters were not trained in using the scoring scheme.

The Spearman-Brown estimate of the split-half reliability for odd and even items is .787 and accepted by the researcher.

Summary

A description of the treatments for Groups E, D1, and D2 was given. The design, scoring, validity, and reliability was discussed for each of the three instruments: Student Inventory of Teacher Behavior, Achievement in Rational Expressions Test, and Problem Solving in Rational Expressions Test. With some qualifications, the instruments were found to have validity and reliability.

Chapter VI

RESULTS

Student Inventory of Teacher Behavior

Hypothesis 1. There is a significant difference between the treatments of any two pairs of Group E, Group D1, and Group D2.

Since the purpose of administering the SITB was to show that the researcher used different treatments with the three groups, a significant difference between class consensus scores should exist. Table 14 shows that all the pairs of treatments were significantly different at the .05 level, and it can be stated that the researcher used different teaching methods in the treatments. To further illustrate the differences between the treatments, Table 15 shows the class consensus scores, and Figure 1 presents them graphically.

TABLE 14. SIGNIFICANCE OF THE DIFFERENCE BETWEEN TWO MEANS FOR INDEPENDENT SAMPLES FOR THE STUDENT INVENTORY OF TEACHER BEHAVIOR

Treatment Pair	df	t-ratio	Level of Significance (p=.05)
E and D1	58	4.075	1.671
E and D2	58	6.312	1.671
D1 and D2	58	2.561	1.671

TABLE 15. CLASS CONSENSUS SCORES FROM THE STUDENT INVENTORY OF TEACHER BEHAVIOR

Category	Group E	Group D1	Group D2
Teacher Omniscience (Items 1 to 5)	23.72	50.42	61.67
Introduction of Generalization (Items 6 to 10)	20.40	61.25	64.16
Control of Pupil Interaction (Items 11 to 15)	54.58	51.67	63.33
Method of Answering Questions (Items 16 to 20)	33.73	59.17	77.08
Uses of Student Responses (Items 21 to 25)	55.40	67.50	87.92
Method of Eliminating False Concepts (Items 26 to 30)	32.91	51.67	68.34

Achievement in Rational Expressions Test

Hypothesis 2. There is no significant difference in the achievement scores of students in A) Groups E and D1, B) Groups E and D2, and C) Groups D1 and D2.

The researcher was interested in the achievement scores to determine whether students taught by discovery would maintain achievement levels equal to students taught by an expository method. Using the Computation for the Analysis of Variation, one-way classification, the F-ratio is .195, and there is no significant difference in the achievement scores of the three groups, as shown in Table 16. To show the effectiveness of the treatments, the mean scores appear in Table 17.

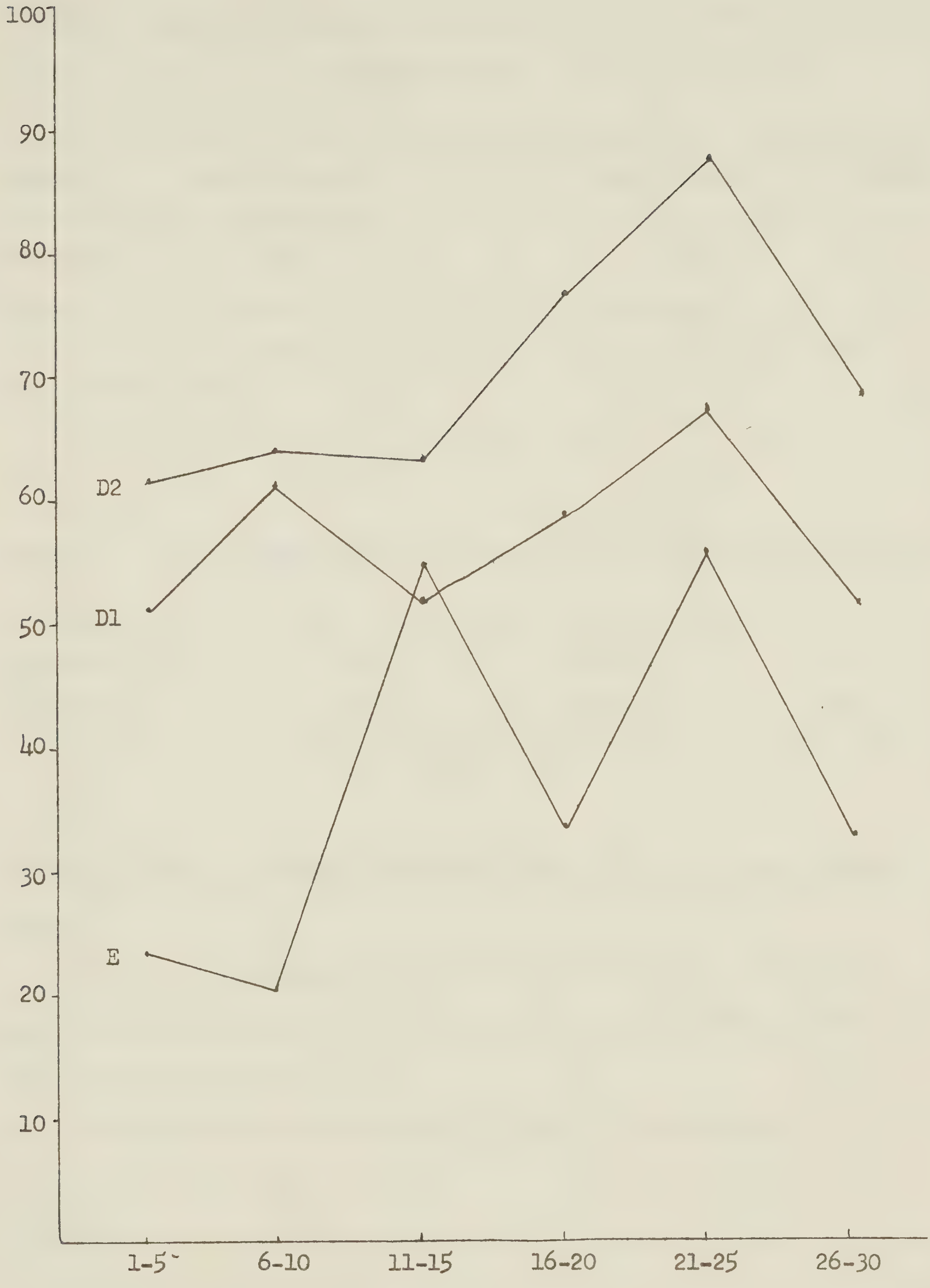

Fig. 1. Profile of the Class Consensus Scores from the Student Inventory of Teacher Behavior

TABLE 16. ANALYSIS OF VARIANCE RESULTS FOR THE ACHIEVEMENT IN RATIONAL EXPRESSIONS TEST

Source	Sums of Squares	df	Mean Squares	F-Ratio
Between	142.89	2	71.445	.195
Within	11347.73	31	366.0558	

$F \geq 3.32$ for $p \leq .05$

TABLE 17. MEAN SCORES FOR THE ACHIEVEMENT IN RATIONAL EXPRESSIONS TEST

Treatment Pair	E and D1	E and D2	D1 and D2
Mean Score	65.45	69.50	65.00

Problem Solving in Rational Expressions Test, Correct Answers Scoring Scheme

Hypothesis 3. There is a significant difference in the problem solving scores of students in A) Groups E and D1, B) Groups E and D2, and C) Groups D1 and D2.

The researcher expected Groups D1 and D2 to have better problem

solving scores than Group E. This was not the case using the Correct Answers scoring scheme. The F-ratio was 1.020 for the Analysis of Variation, one-way classification, as shown in Tables 17 and 18, and there is no significant difference in problem solving mean scores using the Correct Answers scoring scheme.

TABLE 18. ANALYSIS OF VARIANCE RESULTS FOR THE PROBLEM SOLVING IN RATIONAL EXPRESSIONS TEST USING CORRECT ANSWERS SCORING SCHEME

Source	Sums of Squares	df	Mean Squares	F-Ratio
Between	3.863	2	1.9315	1.020
Within	58.707	31	1.894	

$F \geq 3.32$ for $p \leq .05$

TABLE 19. SIGNIFICANCE OF THE DIFFERENCE BETWEEN TWO MEANS FOR INDEPENDENT SAMPLES FOR PROBLEM SOLVING IN RATIONAL EXPRESSIONS TEST, CORRECT ANSWERS SCORING SCHEME

Treatment Pair	df	t-ratio	Level of Significance ($p = .05$)
E and D1	20	.909	1.725
E and D2	21	1.568	1.721
D1 and D2	21	.355	1.721

Problem Solving in Rational Expressions Test, Quality of Answers Scoring Scheme

Hypothesis 3. There is a significant difference in the problem solving scores of students in A) Groups E and D1, B) Groups E and D2, and C) Groups D1 and D2.

For the Quality of Answers scoring scheme, the mean scores for Groups D1 and D2 were not significantly better than the mean score for Group E, as shown in Tables 19 and 20.

TABLE 20. ANALYSIS OF VARIANCE RESULTS FOR THE PROBLEM SOLVING IN RATIONAL EXPRESSIONS TEST USING QUALITY OF ANSWERS SCORING SCHEME

Source	Sums of Squares	df	Mean Squares	F-Ratio
Between	479	2	239.500	.722
Within	10282	31	331.677	

$F \geq 3.32$ for $p \leq .05$

TABLE 21. SIGNIFICANCE OF THE DIFFERENCE BETWEEN TWO MEANS FOR INDEPENDENT SAMPLES FOR PROBLEM SOLVING IN RATIONAL EXPRESSIONS TEST, QUALITY OF ANSWERS SCORING SCHEME

Treatment Pair	df	t-ratio	Level of Significance ($p=.05$)
E and D1	20	.633	1.725
E and D2	21	1.364	1.721
D1 and D2	21	.466	1.721

Problem Solving in Rational Expressions Test, Polya's Four Phases of Problem Solving Scoring Scheme

Hypothesis 3. There is a significant difference in the problem solving scores of students in A) Groups E and D1, B) Groups E and D2, and C) Groups D1 and D2.

The F-ratio of 2.207 from Table 21 does not indicate a significant difference in the mean scores using Polya's Four Phases of Problem Soving scoring scheme. However, when examining pair differences between the means, as in Table 22, there is a significant difference between the mean scores of Groups E and D2 at .05 level.

TABLE 22. ANALYSIS OF VARIANCE RESULTS FOR THE PROBLEM SOLVING IN RATIONAL EXPRESSIONS TEST USING POLYA'S FOUR PHASES OF PROBLEM SOLVING SCORING SCHEME

Source	Sums of Squares	df	Mean Squares	F-Ratio
Between	891.040	2	445.520	2.207
Within	6257.343	31	201.850	

$F \geq 3.32$ for $p \leq .05$

TABLE 23. SIGNIFICANCE OF THE DIFFERENCE BETWEEN TWO MEANS FOR INDEPENDENT SAMPLES FOR PROBLEM SOLVING IN RATIONAL EXPRESSIONS TEST, POLYA'S FOUR PHASES OF PROBLEM SOLVING SCORING SCHEME

Treatment Pair	df	t-ratio	Level of Significance (p=.05)
E and D1	20	.732	1.725
E and D2	21	2.224	1.721
D1 and D2	21	1.395	1.721

Problem Solving in Rational Expressions Test, Quality of Response Approaches Scoring Scheme

Hypothesis 3. There is a significant difference in the problem solving scores of students in A) Groups E and D1, B) Groups E and D2, and C) Groups D1 and D2.

Using the Quality of Response Approaches scoring scheme, the analysis of variance, in Table 23, showed significant differences at .05 level between the three groups. A pair-wise examination of the significance of the difference between the mean scores, in Table 24, indicates a significant difference at .05 level between Groups D1 and D2 in favor of D2 and a significant difference at .01 level between Groups E and D2 in favor of D2.

TABLE 24. ANALYSIS OF VARIANCE RESULTS FOR THE PROBLEM SOLVING IN RATIONAL EXPRESSIONS TEST USING QUALITY OF RESPONSE APPROACHES SCORING SCHEME

Source	Sums of Squares	df	Mean Squares	F-Ratio
Between	103227.3	2	51613.650	4.768
Within	558745.2	31	10824.038	

$F \geq 3.32$ for $p \leq .05$

TABLE 25. SIGNIFICANCE OF THE DIFFERENCE BETWEEN TWO MEANS FOR INDEPENDENT SAMPLES FOR PROBLEM SOLVING IN RATIONAL EXPRESSIONS TEST, QUALITY OF RESPONSE APPROACHES SCORING SCHEME

Treatment Pair	df	t-ratio	Level of Significance ($p=.05$)
E and D1	20	.698	1.725
E and D2	21	2.531	1.721
D1 and D2	21	1.733	1.721

Correlations of Problem Solving in Rational Expressions Test Scores

Since different results were obtained when alternate scoring schemes were used, the correlations between the problem solving scores were found. As shown in Table 25, the pair-wise correlations are well above the critical value of .349 for significance at the .05 level. The two extreme scoring schemes, Correct Answers, which scores only on product, and Quality of Response Approaches, which scores only on process, had the lowest correlation, although still significant.

TABLE 26. CORRELATIONS OF PROBLEM SOLVING IN RATIONAL EXPRESSIONS TEST SCORES

Pairs	Correlations
Correct Answer and Quality of Answer	.870
Quality of Answer and Polya	.916
Polya and Quality of Response Approaches	.869
Quality of Response Approaches and Correct Answer	.659
Correct Answer and Polya	.808
Quality of Answer and Response Approaches	.774

Summary

The three treatment groups were found to be significantly different as judged by the students in the groups. There was no significant difference in the achievement scores of the three treatment groups. There were no significant differences in the problem solving scores of

the three treatment groups using the Correct Answers and Quality of Answers scoring schemes. However, when using Polya's Four Phases of Problem Solving and Quality of Response Apporaches scoring schemes, which evaluate the process of problem solving as well as the product, the problem solving scores were significantly different in favor of Group D2, the group with all aspects of discovery teaching.

Chapter VII

SUMMARY, CONCLUSIONS, AND IMPLICATIONS

Summary

The study is concerned with the effect of discovery teaching in mathematics on achievement and problem solving ability of Grade X matriculation students. Three dimensions of discovery teaching were examined to form three treatments. The interaction dimension involved students working independently or together in groups. Closed interaction, students working independently, was an expository aspect, while open interaction, students working in groups, was a discovery aspect. The guidance dimension involved the teacher control of the classroom learning. Closed guidance, the teacher controlling the classroom learning completely, was an expository aspect, while open guidance, the teacher introducing the problems of study and consolidating the subject matter learned after student pursuit of the problems, was a discovery aspect. The integration dimension involved the exercises or problems students worked on after each lesson or activity. Closed integration, practice exercises similar to those demonstrated by the teacher or hypothesized by the students, was an aspect of expository teaching, while open integration, problems leading to new hypotheses, was an aspect of discovery teaching. The three treatment groups were: 1) Expository (E), containing closed interaction, closed guidance, and closed integration, 2) Discovery I (D1), containing open interaction, open guidance, and closed integration, and 3) Discovery II (D2), containing open interaction, open guidance, and open integration.

Treatments for the three groups were developed by the researcher on the topic, Rational Expressions, from the regular Mathematics 10 curriculum. Thirty six Mathematics 10 students were divided into three treatment groups of twelve students each. All three treatments, taught by the researcher, lasted eleven days, eighty minutes per day, with two days of testing for problem solving and achievement. Problem Solving in Rational Expressions Test was developed by the researcher and Achievement in Rational Expressions Test was developed by the researcher and one of the classroom teachers involved in the experiment. Additionally, students from Groups D1 and D2 were interviewed by the researcher following the experiment. To insure that the treatments were different, the Student Inventory of Teacher Behavior was given to the students at the conclusion of the treatments.

The Student Inventory of Teacher Behavior involved a five-point scale on thirty items testing teacher omniscience, introduction of generalization, control of pupil interaction, methods of answering questions, use of student responses, and method of eliminating false concepts.

The Achievement in Rational Expressions Test contained forty items covering the topics of the unit, Rational Expressions. The items are exercises similar to the practice exercises of Groups E and D1.

The Problem Solving in Rational Expressions Test contained eight items defined as problems which cannot be solved directly using a learned algorithm. The problems involved rational expressions in the solution. Four scoring techniques, Correct Answer, Quality of Answer, Polya's Four Phases of Problem Solving, and Quality of Response Approaches, were applied to the eight items, with separate statistical

analyses reported for each scoring scheme. The Correct Answer scoring scheme awards points only to the correct answer, or product. The Quality of Answers awards full points only to the correct answer, but partial results or errors in a possible solution are also awarded partial points. Polya's Four Phases of Problem Solving looks at all stages of problem solving, understanding the problem, design, procedure, and solution. The Quality of Response Approaches scores on the process of students' problem solving. All approaches to problems are ranked according to feasibility to reach a correct solution and mathematical competence. Scores are awarded according to the ranking. Alternate scoring schemes allow for differing outcomes of problem solving and results of problem solving ability.

Setting and Design Qualifications

Prior to the discussion of the results and conclusions, the setting and limitations of the experiment will be discussed. By setting, the researcher refers to the observations of the researcher while teaching the three groups and the reactions of the students to the experimental treatments. As Wittrock (1966) says:

> In a culture where children are customarily taught by reception rather than discovery, we should not be surprised if their histories are more influential than our brief treatments. Neither should we be surprised if a new discovery procedure is interesting and motivating, at least until the novelty wears thin. (p. 68)

Students generally seemed to cooperate with the researcher, particularly after being told that they would be responsible for the material covered in an achievement test following the treatment and reported to their regular teacher. The prime concern seemed to be the

grade received and the effect of the unit on later work in the course. An outsider entering a situation as this is limited to the topics from the established curriculum to ensure first, the cooperation of the teachers, and second, the cooperation of the students. Another limitation, again to ensure cooperation of students and teachers, was the use of volunteers. Motivations of volunteers are most likely not the same as those of a random sample, but to be able to carry out the research with the cooperation of the school, teachers, and students, volunteers were used.

On the positive side, those students who did volunteer to cooperate in the research seemed willing to try something different. Perhaps the time of year, the last three weeks of April, had something to do with the preference for change. Even students in Group E, who were probably given a stricter treatment than in their regular class, seemed to welcome a change. The smaller class size of twelve, with more individual attention available from the teacher, was also probably a factor in the positive attitudes of the students.

During the treatments, the researcher observed some differences between the groups. The discovery groups were more easily motivated. For instance, Group D2 spent forty minutes arguing over one problem (Problem 19 of Activity 1), with all students actively involved in the discussion, at times getting into a heated debate. The enthusiasm carried over into further discussions in the class. Group D1 also was enthusiastic about the discussions of problem solutions. Many students in Groups D1 and D2 appeared frustrated at times, especially near the beginning of the treatments, but later seemed to be enjoying the discussions, especially when differing viewpoints created arguments.

The students also appeared to enjoy working together in pairs, with only one student (in Group D1) who preferred to work alone, but did cooperate by working in pairs. The researcher's attitude may have also been a factor, but the researcher made a special effort to show equal enthusiasm for all groups.

The students in Group E were attentive during the teacher presentation of the lesson and volunteered answers readily whenever asked by the teacher. They wanted to work together, but were not allowed to, creating some dissension at first. They soon got used to working individually, but apparently their regular teacher allowed them to work together on exercises. They did appreciate the attention of the teacher to answer any questions that arose during the individual work on practice exercises.

One big difference between Group E and Groups D1 and D2 was the dependence on the teacher. Students in Group E grew increasingly more dependent, as determined by the increasing number of questions asked by the students during the teacher explanation session and individual work. Students in Groups D1 and D2 grew more independent when they realized that the teacher was not going to answer any questions about the problems they were working on. However, Group D1 was dependent during the integration phase of each activity, when the teacher did answer questions about the practice exercises they were working on. It is likely that the students in D1 were confused at times with their role in the classroom, having to be alternately independent in the developmtental phase and dependent in the integration phase of the activities. This may have had some effect on their performance on the PSRET.

Another difference between Groups D1 and D2 was in the type of

integration each received. Group D1 practiced the generalizations already made by the class in exercises which were similar to the problems used to make the generalizations. Group D2 worked on problems which used the generalizations made by the class, but which also involved some new generalizations to be discovered. Group D2, then, had more opportunity to work on problems, and then had the additional new generalizations to work with, which neither Group D1 or Group E had been exposed to. Although the items of the PSRET and ARET did not involve these extra generalizations, they still may have had an effect on the mathematical experiences of Group D2, allowing them to perform better on the tests.

At the end of the first week, Groups D1 and D2 seemed to be making faster progress than Group E. The teacher's pacing of the groups probably led to that conclusion, since Group E was totally dependent on the teacher's assignments to begin the next lesson, while Groups D1 and D2 controlled their own time in completetion of the activities. By the end of the treatments, the three groups had made the same progress. Another explanation is that the problems given to Groups D1 and D2 in the first few activities were easier. Group E did not receive those questions until later in the unit.

The researcher noted observations of the students while taking the PSRET. Group E had not been exposed to any problems during the treatments, and the students protested about taking the test. Three students in particular showed extreme frustration while taking the test, and one student left most of the problems unattempted. Another student tried to look at two students' answers, but were stopped. Students often tried asking questions during the test. The group was

very discouraged when handing their tests in. Most students left early, not willing to spend the entire seventy minutes on the test.

The Group D2 students started working immediately on the test. Some said that they did not understand, but were willing to try. A few students asked if they could work together on the test as they had in the treatment, but were refused. The students did not attempt to ask questions of the teacher during the test. All students worked on the test for the entire seventy minutes. While they felt that it was a hard test when they finished, they did not appear to be frustrated.

Group D1 groaned a bit when first receiving the test, protesting that they did not understand. However, they started working soon after without further complaint. They seemed motivated to do the test. They also felt that the test was hard when they handed it in, half of the students handed it in early, and half at the end of the seventy minutes. The students did not appear frustrated.

The biggest difference between Group E and Groups D1 and D2 was the frustration shown during the test. This is probably accounted for in the mind-set created by the treatments. Group E was the most frustrated, never having encountered any problems in the treatments. Group D1 was only slightly frustrated at the beginning of the test, while Group D2 showed no frustration.

The students had interesting comments to make about the treatments. Only students in Groups D1 and D2 were interviewed, since Group E was considered to be similar to the regular classroom treatment. Most students favored the treatment, as explained by Wittrock's novelty idea, the smaller class size, the change in the middle of the term, and

being allowed to work together.

Interviews with Group D2:

Barb: I didn't like it. It seemed easy at first, but you learned afterward that you did it the wrong way. I didn't learn how to do word problems. I liked working in partners.

Karen: You should tell the method first. This way we have to sit and suffer. I got everything except the problems. We always work in partners anyway.

James: It wasn't bad, but I don't like to go into that much detail; the answer is good enough. It was not enjoyable at first, but it picked up. You learn more methods of doing things. You try different methods no matter how crazy it looks. I probably learned better in this class. I learned by doing it on my own. I would recommend it. I put other things I learned in math to use.

Linda: I did not like what you were doing. We didn't get any formulas, didn't know if we were right. We spent too much time on some questions. It was tedious at times. I sort of know what to do on the test. Before the test, in class, I didn't understand while we were doing it, but I studied before the test and understood when you went over it. We had easier problems, then harder ones. We had no formulas and no method of using formulas. The word problems were hard. It was fun though.

Jo-Anne: It was okay. I liked working in groups and learning different methods. I felt I learned as well as in the regular class. The test was hard.

Janice: I couldn't follow a small part of it. I probably learned as much as in the regular class. It was interesting. I liked it when we helped each other, and solved the problems on the board.

Deb: It made you think. I learned as much a I would have in the regular class. Some things were unclear.

Janet: I liked it, it seemed easier. The learning conditions were easier. It was better because we tried our own ideas and where we made mistakes, it was a better way to learn. You can figure it out. It was refreshing from the old way. Excellent and fun.

Susan: Parts of it were better. When you learn by yourself, it sticks in your mind better. I don't remember formulas when given them. This way you make your own formula. Parts of it were too hard.

Garry: I liked it. When doing the problems, I didn't know whether I had good answers. It bothered me.

David: I enjoyed the class. It was fun. More answers are needed.

John: It was dull. It dragged on. We spent too much time on one problem.

Interviews with Group D1:

Marion: It was good except for the behavior of some students. It was too quick on some sections. I learned well on the parts we took time on. I enjoyed the class, good teacher. I'm not sure it would work for the whole class, better for a small group. Good students were having problems.

Ed: I liked it because you didn't have to sit and not say anything. It helped because I did better on the test.

Lori: It would have helped if we would have had different students. It was noisy. I liked to work together. It was a good way to learn because you think more, don't daydream. It keeps you awake.

Gino: I didn't like the noise. It was okay. I learn more when it's quiet.

Dwayne: It was fun. I fooled around but still learned. At times I didn't know what I was doing, but when I figured it out, I really know what I was doing.

Ostap: I didn't understand. I can't factor, so it hurt in this unit. I worked it out, but not the problems we didn't spend much time on. I learned as well as the other method.

Kevin: I liked it. It was easy. Things were explained.

Laurel: I didn't understand parts, it went too fast. I couldn't concentrate. Part was easy. I didn't see too much difference.

Limitations

Since the study was done at only one school and with only one course at one grade level, generalizability is reduced. The assumption is that the high school chosen is representative of the Edmonton Public Schools. It is assumed that each student cooperated and made an honest

effort in answering the problems and exercises. Also, since the researcher is applying the treatments, researcher bias must be taken into account.

In scoring the problem solving tests, only the written work of the students was available, and inferences about their problem solving processes had to be made from the written work.

The researcher was an intruder into the classroom, creating both positive and negative effects.

Summary of Results and Conclusions

Hypothesis 1. There is a significant difference between the treatments of the three Groups E, D1, D2.

The results support this hypothesis, as all pair-wise analyses using the Significance of the Difference Between Two Means are significant at the .05 level.

The conclusion is that the treatments are different as perceived by the students, and Group E students rated the treatment as expository, and Group D1 and D2 students rated the treatments as discovery, with the D2 rating a higher discovery rating than D1.

Hypothesis 2. There is no significant difference in the achievement scores of students in groups E, D1 and D2.

The results support this hypothesis. While expository teaching is often considered to have better results with achievement, the results here indicate that discovery teaching can be as effective in achievement as expository teaching.

Hypothesis 3. There is a significant difference in the problem solving ability of students in Groups E, D1, and D2.

The results for this hypothesis are:

1. Using Correct Answers scoring scheme, there is no significant difference at .05 level in problem solving ability of students in Groups E, D1, and D2.

2. Using Quality of Answers scoring scheme, there is no significant difference at .05 level in problem solving ability of students in Groups E, D1, and D2.

3. Using Polya's Four Phases of Problem Solving scoring scheme, there is a significant difference at .05 level between Groups E and D2, in favor of Group D2. There is no significant difference at .05 level between Groups E and D1 or Groups D1 and D2.

4. Using Quality of Response Approaches scoring scheme, there is a significant difference at .05 level between Groups E and D2 and Groups D1 and D2, each time in favor of Group D2. There is no significant difference at .05 level between Groups E and D1.

5. All correlations of scores from the four problem solving scoring schemes are significant at .05 level, with the lowest correlation between Correct Answers and Quality of Response Approaches, at .659.

The researcher concludes that the scoring schemes, designed to measure different aspects of problem solving, do measure different aspects of problem solving. The Correct Answers method was designed to measure product, while the Quality of Response Approaches was designed to measure process. Since the correlations for these two scoring schemes was the lowest, and these two scoring schemes are designed to measure different aspects, it may be concluded that the researcher has achieved some separation between the measurement of

process and product in problem solving.

Results differ between scoring schemes. It cannot be stated conclusively that discovery teaching improves problem solving ability. However, based on the findings from Quality of Response Approaches, students taught by a discovery approach score significantly higher on problem solving processes than students taught by an expository approach, while scoring equally well on problem solving product. Further, students who have integration problems which involve further discoveries, as Group D2, perform significantly better on problem solving processes than students taught by a discovery approach but given practice exercises involving no new discoveries, as Group D1. The Group D2 students solved only problems leading to new discoveries, possibly improving their problem solving processes. Students in the D1 group may have relied on the exercises at the end of each activity to do most of the learning, rather than concentrating on the problem solving involved in the discovery activity itself.

Polya's Four Phases of Problem Solving also considers process of problem solving, as well as product, and from the results, discovery teaching has a significant effect on problem solving.

When using the discovery approach developed for Group D2, with open interaction, open guidance, and open intergration, the students' problem solving ability in terms of process can be significantly improved, while maintaining problem solving ability in terms of product.

The question remains: The purpose of problem solving is to find the answer, or product. Why should a scoring scheme examine process? What is the relationship of product and process in problem solving? From the Conceptual Framework, Bloom (1950), Le Blanc (1977), Kantowski (1977)

Shulman and Elstein (1975) believe that knowledge of the process by which a problem is solved is at least as important as the product. It is the belief of the researcher that the process of problem solving, which implies openess to new problem situations, willingness to make an attempt at solving problems which look difficult, making statements about problems, and looking for patterns or equations, precedes the attainment of the product, which implies persistence to find the solution. Students must learn means of approaching problems before they can solve them. Through discovery learning, the researcher believes that students become more open to try different problems and develop good approaches to problems. With longer periods of time using a discovery approach, students will also gain in the learning of persistence to find the solution, or product. Process preceeds product, and improvement in problem solving process will lead to improvement in problem solving product.

The students taught by a discovery approach with integration involving problems leading to new discoveries maintain the ability to find the product in problem solving while improving in the process of problem solving. The researcher believes that a longer exposure to a discovery approach will also improve the product of problem solving.

According to Skinner and the laboratory study of learning, the complex process of problem solving obeys the same laws as elementary behavioral processes of stimulus-response behavior. This theory suggested the Quality of Response Approaches, where the approach leads to the product. If indeed, longer treatments of discovery also lead to improved product of problem solving, this theory is supported.

Implications for Practice

At the least, the discovery teaching method is a viable teaching method from the achievement results alone. Teachers have a useful alternative teaching method in discovery teaching, even if achievement is the only goal of the teacher.

Depending on the scoring scheme used to measure problem solving ability, discovery teaching can be used to improve problem solving process. Through the use of discovery teaching, problem solving can be taught while covering prescribed curriculum material of the mathematics program.

A discovery unit was developed by the researcher on a topic not generally considered to be a good discovery topic. It is the belief of the researcher that any topic of the mathematics curriculum can be developed into a discovery unit.

When using a discovery teaching method involving open integration, teachers can expect improved problem solving processes in their students. The product in problem solving will be at least as good as with an expository approach. The results of this study suggest that teachers using a discovery approach with open integration can maintain the achievement and product in problem solving in their students, while adding an improvement in problem solving process.

Implications for Further Study

From the conclusion that product and process of problem solving can be measured separately, the question arises, "Is it reasonable to separate process and product?" There is a need to study more about

the connection or disconnection of product and process in problem solving. There is a need to determine which should be emphasized first, or both equally, in the teaching of problem solving.

In measuring the process of problem solving using the Quality of Response Approaches, inferences were made about the process which students were using to solve problems from their written work. Interviews with students while solving problems or stimulated recall after a problem solving session are methods which could give a closer examination of students' problem solving process. While the interview method has been used to study problem solving processes of students, it has not been used following an experiment with discovery and expository groups.

The researcher has developed four scoring schemes for problem solving and reported the results separately. Could an average be taken for the scoring schemes? What are alternate ways of reporting results from four scoring schemes? Are other scoring schemes possible from attempts to measure different aspects of problem solving?

For this grade level and this topic, the problem solving processes of students were improved using discovery teaching. Further research with other topics and other grade levels needs to be done.

Further research involving a longer treatment period with a discovery group similar to Group D2 could examine the researcher's belief that an extended discovery approach will also improve product in problem solving as well as process.

In comparing this study with Ong's (1976) study, some further questions arise. Ong used an expository approach with Grade VIII students in motion geometry, but gave them integration problems similar

to Group D2, instead of the traditional exercises for practice. Groups 7 and 8 in the Conceptual Framework of this study are similar to Ong's approach. The integration problems differed from Group D2 in that they were divergent, involving several answers, rather than convergent, involving one answer. However, students in both studies had to think of a variety of approaches or processes to solve the problems, so in that way the problems were similar. Ong's experimental students performed significantly better on a convergent problem solving test than his control students, who also were taught by an expository method and practiced traditional exercises.

The common elements of the two treatment groups with significantly better problem solving scores (Group D2 of this study and the experimental group of Ong's study) is not the developmental phase (one being discovery and one expository), but the integration phase. Perhaps the integration phase has more impact on the students, and either discovery or expository teaching in the developmental phase will improve problem solving as long as the integration phase involves problems rather than exercises. The results of Group D1, with practice exercises rather than problems, add support to this conclusion. Further research needs to be done on the same population, independent variables, criterion variables, and topic, comparing Groups 7 and 8 with Group 4 (Group D2) from the Conceptual Framework.

The researcher administered the treatments in this study to avoid the problems of finding teachers willing to learn the treatments and then training the teachers. In this way, the researcher has increased the internal validity of the study by ensuring that the students received the treatments as conceptualized by the researcher, but the

external validity, or generalizability, has been decreased, since the treatments were not used by random classroom teachers. Other researchers have attempted to solve this problem by using programmed materials in the discovery approach, so any random classroom could be used. The treatment then suffers, since difficulties of how to handle open interaction of students and the consolidation and evaluation aspects of the developmental phase. Further research on the best way to administer a discovery teaching method in the classroom with maximum internal and external validity is needed. The researcher believes that the study is acceptable for internal and external validity, and that the results and conclusions made are significant to the field of mathematics education.

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APPENDIX A

TREATMENT FOR GROUP E

Each lesson followed the same pattern. The teacher spent approximiately ten minutes introducing the lesson. The introduction involved definitions, examples, and prescriptions given by the teacher. A discussion period followed in which the teacher worked examples on the board with the students supplying steps when asked to by the teacher. The students also asked any questions they had regarding the lesson. The students were then assigned problems to work individually while the teacher helped individual students with any questions they had. At the end of the period, the problems were corrected by demonstrating the steps in the solution either by the teacher or other students. Each lesson took one eighty-minute period to complete. Some lessons included more than one topic. The topics covered in each lesson are listed.

Lesson 1. Definition of Rational Expressions: Permissible Values

Lesson 2. Simplifying Rational Expressions; Multiplying

Lesson 3. Division of Rational Expressions

Lesson 4. Addition of Rational Expressions

Lesson 5. Subtraction of Rational Expressions

Lesson 6. Division of Polynomials; Dividing by a Monomial; Dividing by a Polynomial

Lesson 7. Rational Expressions in Open Sentences: Equations

Lesson 8. Rational Expressions in Open Sentences: Inequalities

Lesson 9. Word Problems

Lesson 10. Complex Rational Expressions

Lesson 11. Review

TREATMENT FOR GROUP D1

Activity 1. Rational Equalities

The students were given the list of nineteen problems to solve. The only instructions given were to work in pairs to find the value of x. The first eleven problems were given to the students on Day 1, and the rest of the problems were given to the students on Day 2.

Problems

1. $\frac{10}{x} = 2$
2. $\frac{3}{x} + \frac{5}{x} = 4$
3. $\frac{x}{3} + \frac{x}{6} = 12$
4. $\frac{2x}{5} - \frac{3x}{4} = 2$
5. $\frac{x}{4} + \frac{5x}{6} = \frac{2}{3}$
6. $\frac{x - 1}{3} + \frac{2x + 1}{2} = \frac{3}{4}$
7. $\frac{4}{3x} - \frac{2}{x} = 5$
8. $\frac{1}{2x} + \frac{3}{5x} = \frac{1}{2}$
9. $\frac{x+2}{4x} + \frac{2x - 1}{2x} = \frac{5}{6}$
10. $\frac{x - 3}{x} - \frac{3x+2}{3x} = 4$
11. $\frac{x+2}{2} = \frac{x - 1}{3}$
12. $\frac{x - 2}{x} = \frac{x}{x + 3}$
13. $\frac{2}{x} + \frac{3}{x^2} = \frac{1}{2x}$
14. $\frac{4}{x} + \frac{2}{x + 3} = \frac{1}{x^2 + 3x}$
15. $\frac{3}{x + 3} - \frac{2}{x - 3} = \frac{4}{x^2 - 9}$
16. $\frac{5}{x - 2} + \frac{2}{x - 4} = \frac{4}{x^2 - 6x + 8}$
17. $\frac{3}{x} - \frac{2}{x - 1} = \frac{1}{x}$
18. $\frac{x}{2} - 1 = \frac{2 - x}{2}$
19. $\frac{5}{x - 2} = \frac{5}{2 - x}$

Hypotheses

1. Combine (add or subtract) fractions using a common denominator

2. To solve equations involving rational expressions:

   a. Cross-multiply

   b. Guess

   c. Multiply by a common denominator

   d. Combine fractions on one side, then cross-multiply

3. You have to substitute answers back into the equation to find out if you have a zero denominator.

4. Rational expressions with a zero denominator do not fit the definition of rational number: a/b, $b \neq 0$, a,b Integers.

5. Dividing by 0 is impossible.

Practice

1. $\frac{n - 2}{3} + \frac{5}{3} = \frac{2n}{3}$

2. $\frac{2a}{5} + \frac{2a + 1}{5} = 1$

3. $\frac{3}{4} + \frac{n - 4}{8} = \frac{2n - 3}{2}$

4. $\frac{a - 1}{3} + \frac{a + 2}{6} = 2$

5. $\frac{3}{4n^2} + \frac{5}{4n} = \frac{3}{2n}$

6. $\frac{3}{a} + \frac{14}{3a^2} = \frac{2}{3a}$

7. $\frac{5}{4n^2} + \frac{1}{8n} = \frac{3}{2n^2}$

8. $\frac{1}{4a^2} + \frac{5}{6a} = \frac{2}{3a^2}$

9. $\frac{3}{n - 2} + \frac{6}{n^2 - 5n \;\; 6} = \frac{4}{n - 3}$

10. $\frac{7x - 9}{x^2 - 25} - \frac{3}{x - 5} = \frac{2}{x + 5}$

11. $\frac{n + 2}{n} = \frac{n - 1}{n - 5}$

12. $\frac{4}{n - 3} = \frac{4}{3 - n}$

13. $\frac{2a - 3}{2a - 3} = \frac{a - 1}{2a + 3}$

Activity 2. Rational Equalities with Quadratics

The students were instructed to solve the following equations. They could use the hypotheses from Activity 1 when necessary and formulate new hypotheses as needed.

Problems

1. $\frac{2}{n - 3} + \frac{2}{n} = 1$

2. $\frac{3}{a} + \frac{2}{a + 2} = 2$

3. $\frac{n}{2n - 6} - \frac{3}{n^2 - 6n + 9} = \frac{n - 2}{3n - 9}$

4. $\frac{x - 3}{2} = \frac{1}{x - 4}$

5. $\frac{x}{5} = \frac{5}{x}$

6. $\frac{10}{x + 4} - \frac{1}{x} = 1$

7. $\frac{4x - 3}{x - 4} - \frac{2x}{3} = \frac{2x + 5}{x - 4}$

8. $\frac{a + 1}{8} = \frac{2}{a + 1}$

9. $\frac{n - 2}{6} = \frac{a}{a - 2}$

10. $\frac{a - 2}{2a - 1} = \frac{a}{a - 2}$

11. $\frac{n}{4} = \frac{3}{n + 1}$

12. $\frac{2n + 1}{n - 1} - \frac{n}{2} = \frac{n - 4}{n - 1}$

13. $\frac{2n + 5}{n + 2} - \frac{2}{n^2 + 5n + 6} = \frac{n + 5}{n + 3}$

Hypotheses

1. There are two possible answers when there is a square term in the problem.
2. To solve equations with a square term, get all members on one side of the equation so one side equals 0.
3. You can have only one answer when one of the possible solutions makes a zero denominator in an equation with a square term.

Practice

1. $\frac{x}{6} = \frac{6}{x}$

2. $\frac{3}{x + 2} + \frac{2}{x} = 1$

3. $\frac{3n + 1}{n + 4} - \frac{2n}{3} = \frac{n - 3}{n + 4}$

4. $\frac{2a - 1}{a - 3} + \frac{a}{4} = \frac{a + 2}{a - 3}$

5. $\frac{3}{x} + \frac{2}{x - 2} = 1$

6. $\frac{2x - 9}{x - 7} + \frac{x}{2} = \frac{5}{x - 7}$

7. $\frac{x + 1}{2} = \frac{3}{x}$

8. $\frac{3n - 7}{n - 5} + \frac{n}{2} = \frac{8}{n - 5}$

9. $\frac{3a + 2}{a + 3} + \frac{a}{4} = \frac{a - 4}{a + 3}$

10. $\frac{1}{n + 4} = \frac{n - 2}{7}$

11. $\frac{x+5}{4} = \frac{2}{x-2}$

12. $\frac{1}{x} - \frac{x+3}{x+5} = \frac{-30}{x^2+5x}$

Activity 3. Rational Inequalities

Problems

1. $\frac{2x}{3} + \frac{x}{4} \leq x - 1$
2. $\frac{x+1}{2} - \frac{2x+3}{3} > 2x$
3. $\frac{3x-2}{-4} < 3$
4. $\frac{5}{2x} < 1$
5. $\frac{x+3}{2} \geq \frac{2x-2}{5}$
6. $\frac{x+2}{x} < 3$
7. $\frac{2}{x-3} \leq 1$
8. $\frac{x+2}{4} < x - 1$
9. $\frac{2x-5}{-3} < x$
10. $\frac{6}{x} < 2$
11. $\frac{3}{4x} < 2$
12. $\frac{2}{x-4} < 1$

Hypotheses

1. When multiplying both sides of an inequality by a negative, change the sign of the inequality.
2. When you cross-multiply with inequalities, you must be careful which side of the inequality the numbers are on.
3. You can't use cross-multiplying when dealing with inequalities.
4. Short cut in rational expressions with inequalities:
   a. Work out the case with the positive denominator.
   b. Take the negative denominator less than zero for the other part of the answer.
5. You must consider the conditions of the denominator being positive or negative along with the solutions of the inequality.

6. Take the more specific (intersection) of the denominator being positive and the result of the inequality when the sign doesn't change as one answer. Take the more specific (intersection) of the denominator negative and the result of the inequality when the sign changes direction. There are two parts to the solution.

Comments

The students started working the inequalities as they had equations. The results were incomplete or incorrect. The students were satisfied with these answers until the teacher pointed out a member of the solution set which was verified using substitution but not part of the solution set obtained by the students. The students then used substitution to find the solution set until they found patterns. The teacher directed them to problems 3, 6, 7 and 9 for small group work to find the procedures for finding the solution set.

Practice

1. $\frac{n}{2} - 4 > \frac{n}{2}$

2. $\frac{y}{3} - \frac{y}{5} \geq \frac{2}{3}$

3. $\frac{2x}{3} - x \geq 4$

4. $\frac{y+2}{3} - \frac{y+1}{6} \geq 1$

5. $\frac{x+3}{-6} < 4$

6. $\frac{7y+3}{-2} > 9$

7. $\frac{8}{x} > 2$

8. $\frac{5}{2x} < 3$

9. $\frac{2}{x-3} < 1$

10. $\frac{10}{x+1} \geq 5$

11. $\frac{1}{x-1} \leq 3$

12. $\frac{3}{x+4} \leq 1$

Activity 4. Word Problems

Problems

1. Find a number such that the sum of two-thirds of the number and three-fifths of the number is 38.
2. Five-sixths of a number is 6 more than one-half of the number. Find the number.
3. Separate 174 into two numbers whose quotient is 20/9.
4. One number is five times a second number. Find the numbers if the reciprocal of the lesser exceeds the reciprocal of the greater by 2/5.
5. Mr. Benner drove 270 miles in the same amount of time that it took Mr. Strow, traveling 10 miles per hour faster, to travel 330 miles. Find the rate at which Mr. Benner drove.
6. Shawn traveled 12 miles in 2 hours. He walked half the distance and rode his bicycle the remaining distance. If his riding rate was 3 times his walking rate, find the two rates at which he traveled.
7. A bus trip of 180 miles would have taken four-fifths as long if the average speed had been increased by 9 miles per hour. Find the rate at which the bus traveled.
8. A barge travels 36 miles down a river in the same time that it takes to travel 24 miles back up. The currect flows at 3 miles per hour. What is the rate of the barge in still water?
9. Al can paint a house in 6 days. Write in fractions the part completed in 1, 2, 4, 6, and x days.

10. Carol can clean the house in 5 hours. David can do it in 8 hours. Write in fractions the parts done in 1 hour, 2 hours, 3 hours, and x hours, if Carol and David work together.
11. Carson's crew can do the cement work for a new building in 6 days. Green's crew would need 8 days. How many days will it take if the crews work together?
12. Tom can mow his lawn in 5 hours. If Julia helps Tom, the job is done in 2 hours. How many hours would it take Julia working alone?

<u>Hypotheses</u>

Each problem was discussed by the group. A demonstration of different ways of solving each problem was shown and discussed. This method was used rather than general hypotheses to apply to all problems. If equations or inequalities were used, the hypotheses of the previous activities applied.

<u>Practice</u>

1. What number added to both the numerator and denominator of the fraction 4/7 results in a fraction equal to 4/5?
2. One of two positive numbers is twice the other, and their reciprocals differ by 1/10. Find the numbers.
3. A motorist drove 120 miles at a certain rate. On the return trip he doubled his rate. If the round trip required 6 hours, find the rates at which he traveled.
4. Samuel can carpet a floor in 10 hours. If Irene helps him, the job is done in 6 hours. How many hours would it take Irene working alone?

5. Machine A can do a job in 6 hours. Machine B can do the job in 8 hours. How many hours will it take if both machines are working?

6. In a stream that flows at 3 miles per hour, a boy rows 9 miles downstream and then back. His time returning was 3 times that going downstream. Find the rate at which the boy rows in still water.

Activity 5. Complex Expressions

Problems

1. $$\frac{\frac{2}{5} + \frac{1}{4}}{\frac{3}{2} + \frac{7}{10}}$$

2. $$\frac{c - \frac{1}{c}}{\frac{1}{d} + d}$$

3. $$\frac{\frac{2}{5x} + 2}{1 - \frac{3}{10y}}$$

4. $$\frac{\frac{4}{3x^2} + \frac{3}{2x}}{\frac{5}{6x} + \frac{1}{x^2}}$$

5. $$\frac{\frac{4}{x - 2} + \frac{3}{x}}{\frac{5}{x} + \frac{3}{x - 2}}$$

Hypotheses

1. Multiply the numerator and denominator by the common denominator of all fractions within the problem, then simplify.

Practice

1. $$\frac{\frac{5}{x} + \frac{2}{y}}{\frac{3}{x} + \frac{4}{y}}$$

2. $$\frac{a - \frac{1}{4c^2}}{\frac{1}{6c} - 3a}$$

3. $$\frac{\frac{6}{a + 3} - \frac{4}{a - 4}}{\frac{2}{a - 4} + \frac{5}{a + 3}}$$

Activity 6. Dividing by a Polynomial

Problems

1. $(5x^2 - 4x - 12) \div (x - 2)$
2. $(a^3 - 13a - 12) \div (a + 3)$
3. $(14x^2 - 3x + 3x^3 + 7) \div (x + 5)$
4. $(2n^4 - 4n^3 + 7n^2 - 12n + 9) \div (n^2 + 3)$
5. $(3x^4 + 2x^3 - 8x - 48) \div (x^2 - 4)$

Hypotheses

1. Set the problem up like long division of arithmetic.
2. Arrange the divisor and dividend in descending powers, using zero coefficients for any missing powers.

Practice

1. $(y^7 + 1) \div (y + 1)$
2. $(4x^4 - x^2 + 6x + 7) \div (2x + 1)$
3. $(5x - 5x^2 + 4 + 18x^4) \div (2 + 3x)$

TREATMENT FOR GROUP D2

Activity 1. Rational Equalities

The students were given the list on nineteen problems to solve. The only instructions given were to work in pairs to find the value of x. The first eleven problems were given to the students on Day 1, and the rest of the problems were given to the students on Day 2.

Problems

1. $\frac{10}{x} = 2$
2. $\frac{3}{x} + \frac{5}{x} = 4$
3. $\frac{x}{3} + \frac{x}{6} = 12$
4. $\frac{2x}{5} - \frac{3x}{4} = 2$
5. $\frac{x}{4} + \frac{5x}{6} = \frac{2}{3}$
6. $\frac{x - 1}{3} + \frac{2x + 1}{2} = \frac{3}{4}$
7. $\frac{4}{3x} - \frac{2}{x} = 5$
8. $\frac{1}{2x} + \frac{3}{5x} = \frac{1}{2}$
9. $\frac{x + 2}{4x} + \frac{2x - 1}{2x} = \frac{5}{6}$
10. $\frac{x - 3}{x} - \frac{3x + 2}{3x} = 4$
11. $\frac{x + 2}{2} = \frac{x - 1}{3}$
12. $\frac{x - 2}{x} = \frac{x}{x + 3}$
13. $\frac{2}{x} + \frac{3}{x^2} = \frac{1}{2x}$
14. $\frac{4}{x} + \frac{2}{x + 3} = \frac{1}{x^2 + 3x}$
15. $\frac{3}{x + 3} - \frac{2}{x - 3} = \frac{4}{x^2 - 9}$
16. $\frac{5}{x - 2} + \frac{2}{x - 4} = \frac{4}{x^2 - 6x + 8}$
17. $\frac{3}{x} - \frac{2}{x - 1} = \frac{1}{x}$
18. $\frac{x}{2} - 1 = \frac{2 - x}{2}$
19. $\frac{5}{x - 2} = \frac{5}{2 - x}$

Hypotheses

1. Add or subtract terms by finding a common denominator.
2. To solve equations involving rational expressions
   a. Multiply by the common denominator

b. Cross-multiply

c. Combine terms, then cross-multiply

3. The denominator can't be zero, so when a solution makes the denominator zero, the solution is really the empty set.
4. Verify your answers to check for zero denominators.
5. Zero is not the same as undefined.
6. You can do the same thing to both sides of an equation.
7. Two equations are equivalent when the way they are worked out and the solution are equal.

Integration Problems

1. $\frac{2}{n - 3} + \frac{2}{n} = 1$
2. $\frac{3}{a} + \frac{2}{a + 2} = 2$
3. $\frac{3n - 7}{n - 5} + \frac{n}{2} = \frac{8}{n - 5}$
4. $\frac{3a + 2}{a + 3} + \frac{a}{4} = \frac{a - 4}{a + 3}$
5. $\frac{n}{2n - 6} - \frac{3}{n^2 - 6n + 9} = \frac{n - 2}{3n - 9}$
6. $\frac{x - 3}{2} = \frac{1}{x - 4}$
7. $\frac{x}{5} = \frac{5}{x}$

Hypotheses from Integration Problems

1. To solve equations with a square term, get all members on one side, set the equation equal to zero.
2. There are two answers when there is a square term.
3. Check both answers, as one may give a zero denominator.

Activity 2. Rational Inequalities

The students started working the inequalities as they had equations. The results were incomplete or incorrect answers. The students were satisfied with these answers until one student produced a conflicting answer which was correct when substitution was used. Students then

used substitution to find the solution set until they found patterns. The teacher directed them to problems 3, 6, 7, and 9 for small group work to find the procedures for finding the solution set.

Problems

1. $\frac{2x}{3} + \frac{x}{4} \leq x - 1$
2. $\frac{x+1}{2} - \frac{2x+3}{3} > 2x$
3. $\frac{3x-2}{-4} < 3$
4. $\frac{5}{2x} < 1$
5. $\frac{x+3}{2} \geq \frac{2x-2}{5}$
6. $\frac{x+2}{x} < 3$
7. $\frac{2}{x-3} \leq 1$
8. $\frac{x+2}{4} < x - 1$
9. $\frac{2x-5}{-3} < x$
10. $\frac{6}{x} < 2$
11. $\frac{3}{4x} < 2$
12. $\frac{2}{x-4} < 1$

Hypotheses

1. In inequalities, you can't tell if x is positive or negative, so you must have two cases for the denominator positive and negative.
2. The inequality sign does not change when multiplying both sides by a positive, it does change when multiplying both sides by a negative.
3. You must consider the conditions of the denominator being positive or negative along with the solutions of the inequality.
4. Take the more specific (intersection) of 1) the denominator positive and 2) the result of the inequality when the sign doesn't change, as one answer. Take the more specific (intersection) of the denominator negative and the result of the inequality when the sign changes direction. There are two

parts to the solution.

5. Test solutions by substitution.

Integration Problems

1. $\frac{10}{x+4} - \frac{1}{x} < 1$

2. $\frac{4x-3}{x-4} - \frac{2x}{3} > \frac{2x+5}{x-4}$

3. $\frac{a+1}{8} \leq \frac{2}{a+1}$

4. $\frac{n-2}{6} < \frac{2}{n-1}$

5. $\frac{a-2}{2a-1} \geq \frac{a}{a-2}$

6. $\frac{n}{4} < \frac{3}{n+1}$

7. $\frac{2n+1}{n-1} - \frac{n}{2} > \frac{n-4}{n-1}$

8. $\frac{2n+5}{n+2} - \frac{2}{n^2+5n+6} < \frac{n+5}{n+3}$

Hypotheses Generated From Integration Problems

1. You need only the positive case in quadratic inequalities.

Activity 3. Word Problems

Problems

1. Find a number such that the sum of two-thirds of the number and three-fifths of the number is 38.

2. Five-sixths of a number is 6 more than one-half of the number. Find the number.

3. Separate 174 into two numbers whose quotient is 20/9.

4. One number is five times a second number. Find the numbers if the reciprocal of the lesser exceeds the reciprocal of the greater by 2/5.

5. Mr. Benner drove 270 miles in the same amount of time that it took Mr. Strow, traveling 10 miles per hour faster, to travel 330 miles. Find the rate at which Mr. Benner drove.

6. Shawn traveled 12 miles in 2 hours. He walked half the distance and rode his bicycle the remaining distance. If his riding rate was 3 times his walking rate, find the two rates at which he traveled.
7. A bus trip of 180 miles would have taken four-fifths as long if the average speed had been increased by 9 miles per hour. Find the rate at which the bus traveled.
8. A barge travels 36 miles down a river in the same time that it takes to travel 24 miles back up. The current flows at 3 miles per hour. What is the rate of the barge in still water?
9. Al can paint a house in 6 days. Write in fractions the part completed in 1, 2, 4, 6, and x days.
10. Carol can clean the house in 5 hours. David can do it in 8 hours. Write in fractions the parts done in 1 hour, 2 hours, 3 hours, and x hours, if Carol and David work together.
11. Carson's crew can do the cement work for a new building in 6 days. Green's crew would need 8 days. How many days will it take if the crews work together?
12. Tom can mow his lawn in 5 hours. If Julia helps Tom, the job is done in 2 hours. How many hours would it take Julia working alone?

Hypotheses

1. Students remembered the formulae: $d = rt$, $t = d/r$, $r = d/t$
2. Each problem was discussed separately by the group. A demonstration of different ways of solving each problem was shown and discussed.

Integration Problems

1. Machine A can produce 1000 items in 12 hours. Machine B can do this in 10 hours. If machine B starts 2 hours after A has begun, how many hours will it take to produce 1000 items?
2. Charles can paint the garage in 9 hours. Rita can do it in 6 hours. If Charles helps Rita after she has pointed alone for one hour, how many hours will it take to paint the garage?
3. A plane flies with a tail wind of 30 miles per hour for a distance of 2250 miles in five-sixths of the time it takes to fly 2400 miles into a head wind of 20 mph. Find the airspeed of the plane.
4. Fred gave Bill a five-yard head start in a 100-yard dash, and Fred was beaten by one-quarter yard. In how many yards more would Fred have overtaken Bill?

Activity 4. Complex Expressions

Problems

1. $\dfrac{\frac{2}{5}+\frac{1}{4}}{\frac{3}{2}+\frac{7}{10}}$ 2. $\dfrac{c-\frac{1}{c}}{\frac{1}{d}+d}$ 3. $\dfrac{\frac{2}{5x}+2}{1-\frac{3}{10y}}$

Hypotheses

1. Multiply the numerator and denominator by the common denominator of all fractions within the problem, then simplify.

Integration Problems

This activity did not include any integration problems.

Activity 6. Dividing by a Polynomial

Problems

1. $(5x^2 - 4x - 12) \div (x - 2)$
2. $(a^3 - 13a - 12) \div (a + 3)$
3. $(3x^4 + 2x^3 - 8x - 48) \div (x^2 - 4)$
4. $(y^7 + 1) \div (y + 1)$
5. $(4x^4 - x^2 + 6x + 7) \div (2x + 1)$

Hypotheses

1. Set the problem up like long division of arithmetic.
2. Use zero coefficients for any missing powers.

Integration Problems

1. $(14x^2 - 3x + 3x^3 + 7) \div (x + 5)$
2. $(2n^4 - 4n^3 + 9 + 7n^2 - 12n) \div (n^2 + 3)$
3. $(5x - 5x^2 + 4 + 18x^4) \div (2 + 3x)$

Hypotheses for the Integration Problems

1. Arrange the divisor and dividend in descending powers.

APPENDIX B

STUDENT INVENTORY OF TEACHER BEHAVIOR

Block: ____________

Directions: We wish to know some of the kinds of things teachers and students do in mathematics classrooms. In this inventory, each of you will tell how often certain things are done in your math classroom. After your sheets have been scored, they will be destroyed. No one will know what answers you gave. There are no correct or wrong answers.

Read each of the items carefully and fill in the blank the letter of the comment on the answer sheet that most accurately describes how often the event takes place in your math class.

1. When an answer is wrong, our teacher ____ tells us immediately.

   A. Almost Always B. Often C. Sometimes D. Seldom E. Almost Never

2. Our teacher ____ shows us how to solve typical problems.

   A. Almost Always B. Often C. Sometimes D. Seldom E. Almost Never

3. When we ask the teacher how to solve a problem, he ____ shows us.

   A. Almost Always B. Often C. Sometimes D. Seldom E. Almost Never

4. When we ask the teacher how to solve a problem, he ____ gives us only hints that we may use.

   A. Almost Never B. Seldom C. Sometimes D. Often E. Almost Always

5. In taking up work in class, the teacher ____ sees that we get correct answers to all the problems and questions asked.

   A. Almost Always B. Often C. Sometimes D. Seldom E. Almost Never

6. Our teacher ____ explains each new rule before we are given examples to work out.

   A. Almost Always B. Often C. Sometimes D. Seldom E. Almost Never

7. We ____ work on a set of problems without being given any definite ways of working them out.

   A. Almost Never B. Seldom C. Sometimes D. Often E. Almost Always

8. We ____ take up sample problems before we begin to work on a set of exercises.

   A. Almost Always B. Often C. Sometimes D. Seldom E. Almost Never

9. Our teacher ____ encourages us to hypothesize or make guesses at solutions.

   A. Almost Never B. Seldom C. Sometimes D. Often E. Almost Always

10. When we begin discussing work in class we ____ know what rules we will discuss.

    A. Almost Never B. Seldom C. Sometimes D. Often E. Almost Always

11. We seem ____ to be discussing more than one thing at a time in class.

    A. Almost Never B. Seldom C. Sometimes D. Often E. Almost Always

12. We are ____ encouraged to try solving problems even if our method may not work.

    A. Almost Never B. Seldom C. Sometimes D. Often E. Almost Always

13. Our teacher would ____ rather ask a question about the problem than give the correct answer.

    A. Almost Never B. Seldom C. Sometimes D. Often E. Almost Always

14. When one of us works problems out for other class members, our teacher is ____ unhappy.

    A. Almost Never B. Seldom C. Sometimes D. Often E. Almost Always

15. The teacher is ____ willing to discuss any math problems in class even if they are not on the topic.

    A. Almost Never B. Seldom C. Sometimes D. Often E. Almost Always

16. In answering questions in class, our teacher ____ uses his own examples rather than ours.

A. Almost Always B. Often C. Sometimes D. Seldom E. Almost Never

17. In answering my questions, our teacher ____ refers back to the rules rather than what I did.

A. Almost Always B. Often C. Sometimes D. Seldom E. Almost Never

18. In our class, we are ____ given a direct answer to our question or problem.

A. Almost Always B. Often C. Sometimes D. Seldom E. Almost Never

19. Our teacher ____ gives us a rule to use for solving new kinds of problems.

A. Almost Always B. Often C. Sometimes D. Seldom E. Almost Never

20. When we ask questions in class, the teacher ____ would rather that someone from the class answer them.

A. Almost Never B. Seldom C. Sometimes D. Often E. Almost Always

21. We ____ end up with two or three ways of solving the same type of problem.

A. Almost Never B. Seldom C. Sometimes D. Often E. Almost Always

22. Our teacher ____ gives us a chance to try our own method before he points out his method.

A. Almost Never B. Seldom C. Sometimes D. Often E. Almost Always

23. When a student gives a wrong answer, our teacher ____ appears unhappy.

A. Almost Always B. Often C. Sometimes D. Seldom E. Almost Never

24. We are ____ asked to solve problems in only one way.

A. Almost Always B. Often C. Sometimes D. Seldom E. Almost Never

25. I feel that I can solve problems ____ in any way that yields a correct solution.

A. Almost Never B. Seldom C. Sometimes D. Often E. Almost Always

26. When a rule won't apply to all problems, our teacher ____ warns us when not to use it.

A. Almost Always B. Often C. Sometimes D. Seldom E. Almost Never

27. In our assignments we are ____ given problems which cannot be solved using rules discussed in class.

A. Almost Never B. Seldom C. Sometimes D. Often E. Almost Always

28. When a new problem comes up, our teacher ____ shows us exactly which rules to use.

A. Almost Always B. Often C. Sometimes D. Seldom E. Almost Never

29. Our teacher ____ lets us try for ourselves to use a rule on a problem even when he knows it does not work.

A. Almost Never B. Seldom C. Sometimes D. Often E. Almost Always

30. We are ____ cautioned to think through our problems and solutions carefully.

A. Almost Always B. Often C. Sometimes D. Seldom E. Almost Never

ACHIEVEMENT IN RATIONAL EXPRESSIONS TEST

Name ______________________________ Block ______

1. Which of the following are rational expressions if $x = 2$ and $y = 1$? Circle your answers.

   (a) $\frac{x-3}{x-1}$ (b) 3 (c) x (d) $\sqrt{x^2+3}$ (e) $\frac{x+2}{y-1}$

   (f) $\frac{x^2+3x+2}{y^2+5x-14}$ (g) $\frac{y^2+4y+2}{x^2-4}$ (h) $\frac{x^2-4x+4}{y+1}$

2. Gives the values which are <u>not</u> permissable replacements for the variables in the following?

   (a) $\frac{7c}{x^2+3x-28}$ ______________ (b) $\frac{x^2+2}{x^2-2}$ ______________

   (c) $3n(n-1)^{-1}$ ______________ (d) $\frac{c+d}{c^2-cd}$ ______________

   (e) $\frac{x^4-4}{(x^2+5x+6)(3x-6)}$ ______________________________

<u>Simplify</u>

3. $\frac{8m^2}{16m^3} =$ ______________________________

4. $\frac{14}{7y-7} =$ ______________________________

5. $\frac{5mn^2p^3}{-15\ m^3n^2p} =$ ______________________________

6. $\frac{3x+4}{9x^2-16} =$ ______________________________

7. $\dfrac{y^2 - y - 12}{y^2 + y - 20} =$ ______________________

8. $\dfrac{px + qx + py + qy}{px - qx + py - qy} =$ ______________________

9. $\dfrac{3m-3p}{6} \times \dfrac{2a^2 + 4ab + 2b^2}{m^2 - p^2} =$ ______________________

10. $\dfrac{x^2 + 7x + 10}{x^2 - 2x - 15} \times \dfrac{x^2 - 25}{4 - x^2} \times \dfrac{x+2}{x+5} =$ ______________________

11. $\dfrac{3p+9}{5y+10} \div \dfrac{6}{25} =$ ______________________

12. $\dfrac{c+d}{c^2 - cd} \div \dfrac{1}{d^2 - c^2} =$ ______________________

13. $\dfrac{x^2 - 9}{a^2 - b^2} \div \dfrac{x^2 - 6x + 9}{a^2 + 2ab + b^2} =$ ______________________

14. $\dfrac{x^2 - 3x - 28}{x^2 - 16} \times \dfrac{x^2 - 2x - 8}{x^2 + 2x - 8} \div \dfrac{x^2 - 16x + 63}{x^2 + 4x} =$ ______________________

15. $\dfrac{a}{a^2 - 4} + \dfrac{2}{a^2 - 4} =$ ______________________

16. $\dfrac{a+b}{ab} - \dfrac{a-c}{bc} =$ ______________________

17. $\dfrac{a-b}{5} - \dfrac{a+b}{7} =$ ______________________

18. $\dfrac{x+1}{x+2} + \dfrac{x+3}{x+4} =$ ______________________

19. $\dfrac{a-1}{a-2} - \dfrac{a-3}{a-4} =$ ______________________

20. $\dfrac{4k}{3(k-5)} - \dfrac{3k}{2(5-k)} =$ ______________________

21. $\dfrac{3x}{x^2-49y^2} - \dfrac{2x}{x+7y} =$ ____________________

22. $\dfrac{5}{x^2-8x+15} - \dfrac{3}{x-5} + \dfrac{2}{3+x} =$ ____________________

23. $\dfrac{2}{a^2-6a-7} - \dfrac{3}{a+1} + \dfrac{3}{a-7} =$ ____________________

24. $\dfrac{6a}{a-3b} + \dfrac{4b}{2a-b} - \dfrac{2ab}{2a^2-7ab+3b^2} =$ ____________________

25. $\dfrac{1}{x^2+3x+2} + \dfrac{2}{x^2+5x+6} + \dfrac{3}{x^2+4x+3} =$ ____________________

26. $\dfrac{1}{xy-3x-3y+9} + \dfrac{1}{3-y} - \dfrac{1}{x-3} =$ ____________________

Divide:

27. $(6x^3 - 2x^2 - 2x + 30) \div (3x+5)$

28. $(3x^2 - 5x - 30) \div (x - 4)$

29. $(10x^3 - 34x + 8) \div (2x+4)$

Solve the Following:

30. $\dfrac{x}{4} + \dfrac{7}{12} = \dfrac{x+1}{3}$ ____________________

31. $\dfrac{3}{4a} - \dfrac{5}{6a^2} = \dfrac{1}{3a}$ ____________________

32. $\dfrac{5}{x-3} + \dfrac{6}{9-x^2} = \dfrac{4}{x+3}$ ____________________

33. $\frac{3}{x+1} = \frac{2}{x}$ ____________________

34. $\frac{x+1}{2} = \frac{3}{x}$ ____________________

35. $\frac{2x-9}{x-7} + \frac{x}{2} = \frac{5}{x-7}$ ____________________

36. $\frac{n-3}{8} - \frac{n+2}{3} > \frac{5}{12}$ ____________________

37. $\frac{2t-5}{3} > 1$ ____________________

38. $\frac{x+2}{x} < 3$ ____________________

39. Paul can plant his wheat crop in ten days. His daughter can do it in fifteen days. How many days will it take if they work together?

40. A motor boat travels 25 miles downstream in the same amount of time it takes to travel 15 miles upstream. The current flows at 5 miles per hour. Find the rate of the motorboat in still water.

PROBLEM SOLVING IN RATIONAL EXPRESSIONS TEST

Name ______________________________ Block________

1. Find two rational expressions whose sum is $\dfrac{7x+9}{(x+1)(x+2)}$.

2. Express as the sum of three fractions: $\dfrac{4x^2+2}{x(x-1)(x+2)}$

3. Two cars race on a 4-mile oval track. The sum of the rates at which they travel is 200 miles per hour. Find the rate of each if the faster car gains one lap in 40 minutes.

4. At what time between 3 and 4 o'clock will the hands of a clock be together? At what time will they be opposite each other?

5. A job can be done by 8 men in 3 hours or by 15 boys in 5 hours. How long would it take 3 men and 25 boys together?

6. A man contracts to build a road in 72 days, a job requiring 60 men. The man hires 50 men who work for a while until he realizes that he must hire 30 more to finish on time. How many days do these 30 men work?

7. The digits of a three-digit number are three consecutive integers. The middle digit is the greatest and the first digit is the least. If the number is divided by the sum of its digits, the quotient is 229/7. Find the number.

8. With certain fractions, you can obtain interesting patterns by breaking the numerator into two equal factors and at the same time breaking the denominator into an indicated sum. For example:

$$121 = \frac{484}{4} = \frac{22 \times 22}{1+2+1}$$

$$12321 = \frac{333 \times 333}{1+2+3+2+1}$$

What is the value of $\dfrac{4444 \times 4444}{1+2+3+4+3+2+1}$?

Using the pattern, what is a fraction for 1,234,567,654,321 ?

APPENDIX C

TABLE 27. CATEGORIES OF RESPONSES FOR PROBLEM 1 OF THE PSRET

Category	Description	Score
1.	No response	0
2.	Repeated the question	10
3.	Multiplied the terms in the denominator	20
4.	Separated the numerator and denominator as given, ie. $\frac{7x}{x+1} + \frac{9}{x+2}$	30
5.	Used complex fractions, ie. $\frac{\frac{7x+9}{x+2}}{x+1}$	40
6.	Correct answer by guessing	50
7.	Made an equation using $\frac{7x+9}{(x+1)(x+2)}$	60
8.	Separated the denominator, as $\frac{}{x+1} + \frac{}{x+2}$ and equated the numerator and denominator	70
9.	Used the denominator $(x+1)(x+2)$ and separated the numerator, ie. $\frac{x}{(x+1)(x+2)} + \frac{6x+9}{(x+1)(x+2)}$	80
10.	Took half of $\frac{7x+9}{(x+1)(x+2)}$	90
11.	Separated the numerator into factorable parts ie. $\frac{3x+3}{(x+1)(x+2)} + \frac{2x+4}{(x+1)(x+2)} + \frac{2x+2}{(x+1)(x+2)}$	100

TABLE 28. CATEGORIES OF RESPONSES FOR PROBLEM 2 OF THE PSRET

Category	Description	Score
1.	No response	0
2.	Separated the numerator and denominator as given ie. $\frac{1}{x} + \frac{4x^2}{(x-1)} + \frac{2}{(x+2)}$	10
3.	Multiplied the denominator, then separated it ie. $\frac{1}{x^3} + \frac{4x^2}{x^2} - \frac{2}{2x}$	20
4.	Added additional terms keeping the numerator and denominator as given, ie. $\frac{4x^2+2}{1} + \frac{0}{0} + \frac{1}{x(x-1)(x+2)}$	30
5.	Factored the numerator	40
6.	Used complex fractions, ie. $\frac{\frac{2x^2}{(x-1)(x+2)}}{x} + \frac{\frac{2x^2}{x(x+2)}}{(x-1)} + \frac{\frac{2}{x(x-1)}}{(x+2)}$	50
7.	Correct answer by guessing	60
8.	Used the denominator $x(x-1)(x+2)$, separated the numerator ie. $\frac{x^2+1}{x(x-1)(x+2)} + \frac{2x^2+1}{x(x-1)(x+2)} + \frac{x^2}{x(x-1)(x+2)}$	70
9.	Divide $\frac{x^2+1}{x(x-1)(x+2)}$ by three	80
10.	Separated the denominator, as $\frac{}{x} + \frac{}{(x-1)} + \frac{}{(x+2)}$, guessed at the numerators, then verified by multiplying	90
11.	Separated the numerator into factorable parts ie. $\frac{2x^2+4x}{x(x-1)(x+2)}$ $\frac{-x^2-x+2}{x(x-1)(x+2)}$ $\frac{3x^2-3x}{x(x-1)(x+2)}$	100

TABLE 29. CATEGORIES OF RESPONSES FOR PROBLEM 3 OF THE PSRET

Category	Description	Score
1.	No response	0
2.	Guess	7
3.	Incorrect formula	13
4.	Correct formula, $d = rt$, substituted values from the problem directly into the formula, ie. $d = 4$, $r = 200$, $t = 40$	20
5.	Correct formula, d rt, substituted values from the problem directly into the formula, changed 40 minutes to hours 40/60	27
6.	Used a proportion, ie. $\frac{1}{x} : : \frac{x}{200}$, $\frac{4}{x} = \frac{1}{200}$	33
7.	Correct formula, $d = 4$, $r = 201\text{-}x$, $t = 40$	40
8.	Correct formula with a diagram	47
9.	Number of laps is $200/4 = 50$ to get miles per hour	53
10.	Changed 40 minutes to hours and used sum of rates is 200	60
11.	Correct formula, sum of rates expressed as $\frac{4}{x+40} + \frac{4}{x}$, used an equation.	67
12.	Combination of Categories 4 and 11, used two methods	73
13.	Combination of Categories 6 and 11	80
14.	Made a chart for rate, distance, time	87
15.	Used the correct formula with minutes changed to hours, made an equation using rate per lap and set times equal	93
16.	Changed minutes to hours, calculated the difference in speed, ie. $4(2/3) = 6$ mph, made an equation using the sum of the rates equalling 200	100

TABLE 30. CATEGORIES OF RESPONSES FOR PROBLEM 4 OF THE PSRET

Category	Description	Score
1.	Guess with no work shown	20
2.	Drew a picture of a clock	40
3.	Used rates of travil of minute and hour hands	60
4.	Used rates of travel of minute and hour hands with a diagram	80
5.	Made a table of times for minute and hour hands	100

TABLE 31. CATEGORIES OF RESPONSES FOR PROBLEM 5 OF THE PSRET

Category	Description	Score
1.	No response	0
2.	Guess	6
3.	Expression $3/8 + 2/25$, then inverted	13
4.	Equation $8/3 + 15/5 = x$	19
5.	Equation $1/3 + 1/5 = 8/13$	25
6.	Equation $3/8 + 1/3 = x/3 + x/25$	31
7.	Equation $8/3 + 15/5 = 3/x + 25/x$	38
8.	Proportions, ie. $\frac{15}{5} :: \frac{25}{x}$, $\frac{3}{n} = \frac{25}{n}$	44
9.	Used deduction: 8 men-3 hours, 3 men-3 x hours 15 boys-5 hours, 25 boys-5-x hours	50
10.	Used deduction: 1 man in 3/8 hour, 1 boy in 1/3 hour Then tried equations: $5x(3) = 15y(5)$	56
11.	Equation $\frac{3}{x} + \frac{25}{3x} = 1$	63
12.	Equation $\frac{3x}{64/3} + \frac{5x}{9} = 1$	69
13.	Equation $\frac{3}{x} + \frac{15+10}{x} = 1$	75
14.	Equation $\frac{x}{3/8} + \frac{x}{25/15} = 1$	81
15.	Equation $x(8/3 + 15/5) = 1$ and $\frac{x}{8} + \frac{x}{10/3} = 1$	88
16.	Chart: $\begin{array}{lcccc} \text{men} & 8 & 4 & 2 & 3 \\ \text{hours} & 3 & 6 & 12 & 9 \end{array}$ $\quad \begin{array}{lccc} \text{boys} & 15 & 30 & 25 \\ & 5 & 2\frac{1}{2} & 2\frac{1}{2} \end{array}$	94
17.	Equation: $\frac{3x}{24} + \frac{8x}{24} = \frac{24}{24} \quad \rightarrow \quad x = \frac{24}{11}$	100

TABLE 32. CATEGORIES OF RESPONSES FOR PROBLEM 6 OF THE PSRET

Category	Description	Score
1.	No response	0
2.	Guess	5
3.	Restating the problem	10
4.	Ratios, ie. 72/60, 72/50, 180/5, 300/5	15
5.	Equation: $60+72=50+72-x+30-x$	20
6.	Equation: $\frac{72}{60}=\frac{50}{x}+\frac{30}{x}$	25
7.	Equation: $72-((72\div 2)+(36\div 2))=18$	30
8.	Equation: $\frac{72}{60}+\frac{50}{x}=30x$	35
9.	Equation: $50x+80y=72$	40
10.	Equation $\left[\frac{(72-60)3/5}{72}\right]30=3$	45
11.	Formula: job = time x men, Equation: $x=50/72+80/72$	50
12.	Equation: $\frac{72}{60}=\frac{x+30}{50}$	55
13.	Equation: $\frac{50}{x}+\frac{30}{72-x}=\frac{60}{72}$	60
14.	Equation: $\frac{60}{72}=\frac{80x}{72}$	65
15.	Equation: $\frac{50}{72-x}+\frac{80}{x}=\frac{60}{72}$	70
16.	Proportions: $\frac{60}{72}=\frac{50}{x}$, $\frac{60}{72}=\frac{80}{x}$	75
17.	Equation: $(50+30)(72+x)=1$	80
18.	Equation: $(50-x)/60=1$	85
19.	Equations: 30 men = 144, 50 men = 86.5, $\frac{x}{86.5}+\frac{x}{144}=\frac{72}{72}$	90

TABLE 32. (Con'd)

Category	Description	Score
20. Equation:	$\frac{x}{50}+\frac{x}{30}=1$	95
21. Deduction:	$72 \times 60 = 4320$ man days $72 \times 50 = 3600$ man days $4320 - 3600 = 720 \rightarrow 720/30 = 24$	100

TABLE 33. CATEGORIES OF RESPONSES FOR PROBLEM 7 OF THE PSRET

Category	Description	Score
1.	No response	0
2.	Guess	6
3.	Equation: $\frac{- - -}{1} = x$	11
4.	Contends that the problem cannot be solved since three consecutive integers cannot have the middle number largest	17
5.	Ratio: $229/7 = 33$	22
6.	Ratio: $\frac{x}{229/7}$	28
7.	Deduction: $x + 2x + 3x$, $x = \sqrt{229/7}$	33
8.	Equation: $\frac{x}{1} + \frac{x+1}{1} + \frac{x+2}{1} = \frac{229}{7}$	39
9.	Equation: $1x + 2x + 3x = \_\ \_\ \_ = 229/7$	44
10.	Expression: $\frac{(x+1) + (x+2) + (x+3)}{x}$	50
11.	Expression: $(3x+3) \cdot 229/7$	56
12.	Expression: $\frac{229 \times 3}{7 \times 3} = \frac{687}{21}$	61
13.	Equation: $\frac{xyz}{x+y+z} = \frac{229}{7}$ No place value	67
14.	Equation: $\frac{x(x+2)(x+1)}{3x+3} = \frac{229}{7}$ No place value	72
15.	Equation: $229/7\ (x + x + 1 + x + 2) = x$	78
16.	Equation: $\frac{x}{(x+1)+(x+2)+(x+3)} = \frac{229}{7}$	83
17.	Table of possible values, trial and error: 123,234,345,456,567,678,789	89
18.	Equation: $\frac{abc}{a+b+c} = \frac{229}{7}$, $a + b + c < 24$, trial and error	94

TABLE 33. (Con'd)

Category	Description	Score
19.	Equation: $\frac{100(x-1)+10(x+1)+x}{3x} = \frac{229}{7}$	100

TABLE 34. CATEGORIES OF RESPONSES FOR PROBLEM 8 OF THE PSRET

Category	Description	Score
1.	No response	0
2.	Computation in both parts	14
3.	Computation in first part, no answer in second part	29
4.	No response first part, pattern second part	43
5.	Computation first part, pattern second part, verified by computation	57
6.	Computation first part, pattern second part	71
7.	Made a table showing the pattern	86
8.	Pattern both parts	100